New Integrable Hierarchies from Vertex Operator Representations of Polynomial Lie Algebras

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Abstract

We give a representation–theoretic interpretation of recent discovered coupled soliton equations using vertex operators construction of affinization of not simple but quadratic Lie algebras. In this setup we are able to obtain new integrable hierarchies coupled to each Drinfeld–Sokolov of A, B, C, D hierarchies and to construct their soliton solutions.

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1 Introduction

One of the most important achievement of the representations of the affine Lie algebras and their groups is surely the Lie theoretical explanation of the Hirota [9] bilinear approach to the soliton equations. This beautiful piece of mathematics is the result of an important sequence of relevant papers, which starts in 1981 with the works of Sato [23] [25], where the link between the soliton equations and the infinite–dimensional groups has been brought to the light for the first time. Some years later Date, Jimbo, Kashiwara and Miwa gave a construction of the Kadomtsev–Petviashvili (KP) and Korteweg–de Vries (KdV) hierarchies in terms of the vertex operators representating the affine lie algebra $\mathfrak{a}_\infty$ and $\widehat{\mathfrak{sl}}_2$ respectively, while Segal and Wilson [26] have examinated the same equation from a geometrical point of view. Finally Drinfeld, Sokolov [5], Kac Peterson and Wakimoto [12] [15][14] have extended this theory to all affine Lie algebras. These results have suggested to find a similar interpretation for others hierarchies of soliton equations, for example in a recent work [3] Billig has obtained this goal for the sine–Gordon.

The main aim of the present paper is to contribute to the research in this direction. Our starting point is the following "coupled KdV equations" which appears in many very recent papers of different authors like Hirota, Hu, Tang, [10], Sakovich [22], Kakei [16]:

$$\begin{aligned} v_t + 6vv_x + v_{xxx} &= 0 \\ w_t + 6vw_x + w_{xxx} &= 0. \end{aligned} \tag{1.1}$$

The corresponding bilinear Hirota form of these equations (and actually of many others closely related, among them "coupled KP equations" [22]) is namely known [10] together with some soliton solutions, but, as far as we know, it is still missing their broader Lie–theoretic interpretation. In this paper we shall show how these equations are a particular case of a very wide class of "coupled soliton equations" which can be obtained using the vertex operator realization of a new class of infinite dimensional Lie algebras. These latter algebras are the affinization of not simple finite dimensional Lie algebras, which still posses a symmetric non degenerated ad–invariant bilinear form. Therefore in our long journey towards our task we shall be enforced to develop a vertex operator algebras theory for a class of Lie algebras which are not the the affinization of semisimple ones. Nevertheless our construction will allow us to produce coupled soliton equations corresponding to each of the Drinfeld–Sokolov and the AKP BKP CKP DKP hierarchies, although for sake of brevity only the case of the coupled AKP BKP and their reductions to opportune generalisations of the affine Lie algebras $A_1^{(1)}$ $A_2^{(1)}$, $A_1^{(2)}$ and and $B_2^{(1)}$ are explicitly examined in the paper. These reductions in turn provide a clear explanation of how the coupled KP equations become the

coupled KdV ones by neglecting the dependency from one particular variable. Finally the action on the space of representation of the corresponding infinite dimensional groups will provide, exactly as in the usual case, a class of multi–soliton solutions.

The paper is organised as follows: in the second section we shall describe a class of finite dimensional Lie algebras known in the literature as polynomial Lie algebras [17],[21] which, roughly speaking, can be regarded as direct sum of semisimple Lie algebras endowed with a non canonical Lie bracket. We shall show that these Lie algebras can be constructed in completely different ways: namely as particular finite dimensional quotients of an infinite dimensional algebra and as a Wigner contraction of a direct sum of finite dimensional semisimple Lie algebras, or finally as tensor product between a finite dimensional Lie algebra $\mathfrak{g}$ and a nilpotent commutative ring. Further we shall show how on these Lie algebras is defined a class of symmetric non degenerated ad–invariant bilinear forms if a such bilinear form exits on $\mathfrak{g}$. In the next section it will be shown how these latter bilinear forms can be used to affinize those in general non simple Lie algebras. Then in section 4 their vertex operator algebras construction is presented. Once this result is achieved we can tackle the problem to construct the corresponding generalised Hirota bilinear equation and their multisoliton solution in term of τ–functions. This will be done in the fifth and last section where further the case of the coupled AKP BKP and their reduction to Lie algebras generalising the algebras $A_1^{(1)}$, $A_2^{(1)}$, $A_1^{(2)}$, and $B_2^{(1)}$ are presented into details.

2 The polynomial Lie algebras

The aim of this first section is to construct in a few different ways a class of finite dimensional Lie algebras (called in what follows polynomial Lie algebras to keep the name usually used in the literature, see for example [17] and [21]) which are going to play a crucial role in the whole paper. The reader will surely recognise that actually our constructions can be straightforward generalised to the realm of the infinite dimensional Lie algebras; but, although we shall need for our porpoises this generalisation, let us here for sake of concreteness restrict ourself to the maybe simpler finite dimensional case.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let us denote by $\mathbb{C}(\lambda)$ the ring of the polynomials in a complex variable λ with coefficients in $\mathbb{C}$:

$$\mathbb{C}(\lambda) = \{\sum_{i=0}^{k} c_i \lambda^i |\ k \in \mathbb{N}\}. \tag{2.1}$$

The space $\mathbb{C}(\lambda)$ of course inherits form its associative and commutative product an abelian (i.e. trivial) Lie algebra structure. Therefore we can regard the space

$$\mathfrak{g}(\lambda) = \mathfrak{g} \otimes \mathbb{C}(\lambda) \tag{2.2}$$

as a Lie algebra tensor product. It may be identified with the Lie algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$, hence an element $X(\lambda)$ in $\mathfrak{g}(\lambda)$ can be viewed as the mapping $X : \mathbb{C} \to \mathfrak{g}$, $X(\lambda) = \sum_{k=0}^{\infty} X_k \lambda^k$ where $X_k = 0$ for all but a finite number of indices. In this setting the Lie bracket of two elements in $\mathfrak{g}(\lambda)$, $X(\lambda) = \sum_{k=0}^{\infty} X_k \lambda^k$ and $Y(\lambda) = \sum_{k=0}^{\infty} Y_k \lambda^k$ can be written explicitly as

$$[X(\lambda), Y(\lambda)] = \sum_{n=0}^{\infty} (\sum_{j=0}^{n} [X_j, Y_{n-j}]_{\mathfrak{g}}) \lambda^n \tag{2.3}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie bracket defined on $\mathfrak{g}$.

If on $\mathfrak{g}$ is defined an ad–invariant bilinear form $\langle \cdot, \cdot \rangle$ (for instance the Killing form) then we can carry it to a bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on $\mathfrak{g}(\lambda)$ by setting on $\mathbb{C}(\lambda)$ the standard inner product

$$(p(\lambda), q(\lambda)) = (\sum_{i=0}^{\infty} p_i \lambda^i, \sum_{j=0}^{\infty} q_j \lambda^j) = \int_{|\lambda|=1} \lambda^{-1} p(\lambda) \overline{q(\lambda)} d\lambda = \sum_{i=0}^{\infty} p_i \overline{q_i} \tag{2.4}$$

(where $\overline{z}$ denotes the complex conjugate of z and one has to keep in mind that all the sum are on a finite set and that on the unit circle both $p(\lambda)$ and $p(\lambda^{-1})$ are well defined). More precisely on $\mathfrak{g}(\lambda)$ will be defined the bilinear form:

$$\begin{aligned} \langle X(\lambda), Y(\lambda) \rangle_\lambda &= \langle \sum_{i=0}^{\infty} X_i \lambda^i, \sum_{j=0}^{\infty} Y_j \lambda^j \rangle_\lambda \\ &= \int_{|\lambda|=1} \lambda^{-1} \langle X(\lambda), Y(\lambda^{-1}) \rangle d\lambda = \sum_{i=0}^{\infty} \langle X_i, Y_i \rangle. \end{aligned} \tag{2.5}$$

Unfortunately this bilinear form, while it is not degenerate if the bilinear form on $\mathfrak{g}$ is not degenerate, turns out to be in general not ad–invariant even if the form chosen on $\mathfrak{g}$ (like in the case of the Killing form) is ad–invariant. Suppose indeed $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ endowed with the Killing form and set

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then

$$\langle H, [X\lambda, Y\lambda] \rangle_\lambda = \langle H, H\lambda^2 \rangle_\lambda = 0$$

but

$$\langle [H, X\lambda], Y\lambda \rangle_\lambda = \langle 2X\lambda, Y\lambda \rangle_\lambda = 2 \neq 0.$$

On the other hand there exist on $\mathfrak{g}(\lambda)$ non trivial symmetric ad–invariant form, which are surely more exotic than that previously considered and moreover degenerated. Nevertheless these latter bilinear forms are still worth to be considered,

because from them one can derive symmetric ad–invariant bilinear forms on opportune quotients of $\mathfrak{g}(\lambda)$ which will turn out to be actually non degenerated.

Definition 2.1 *For any symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and for any sequence $\mathcal{A}=\{a_n\}_{n\in\mathbb{N}}$ of complex numbers a_k, let $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ be the bilinear form on $\mathfrak{g}(\lambda)$ given by the formula:*

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\mathcal{A}}:\mathfrak{g}(\lambda)\times\mathfrak{g}(\lambda) &\longrightarrow \mathbb{C}\\ X(\lambda),Y(\lambda)\in\mathfrak{g}(\lambda) &\longrightarrow \langle X(\lambda),Y(\lambda)\rangle_{\mathcal{A}}=\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\langle X_{n-k},Y_k\rangle_{\mathfrak{g}}\end{aligned}\tag{2.6}$$

It easily checked the following:

Proposition 2.2 *If the symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ defined on $\mathfrak{g}$ is ad–invariant then the same is also true for the bilinear form (2.6) defined on $\mathfrak{g}(\lambda)$.*

Proof. We have to prove that for any elements $X(\lambda)=\sum_{n=0}^{\infty}X_n\lambda^n$, $Y(\lambda)=\sum_{k=0}^{\infty}Y_j\lambda^j$, $Z(\lambda)=\sum_{k=0}^{\infty}Z_k\lambda^k$ in $\mathfrak{g}(\lambda)$ it holds that

$$\langle[X(\lambda),Y(\lambda)],Z(\lambda)\rangle_{\mathcal{A}}=\langle X(\lambda),[Y(\lambda),Z(\lambda)]\rangle_{\mathcal{A}}.\tag{2.7}$$

Now

$$\begin{aligned}\langle[X(\lambda),Y(\lambda)],Z(\lambda)\rangle_{\mathcal{A}} &= \sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\langle[X,Y]_{n-k},Z_k\rangle_{\mathfrak{g}}\\ &= \sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\langle\sum_{j=0}^{n-k}[X_j,Y_{n-k-j}],Z_k\rangle_{\mathfrak{g}}\end{aligned}\tag{2.8}$$

while

$$\begin{aligned}\langle X(\lambda),[Y(\lambda),Z(\lambda)]\rangle_{\mathcal{A}} &= \sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\langle X_{n-k},[Y,Z]_k\rangle_{\mathfrak{g}}\\ &= \sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\langle X_{n-k},\sum_{j=0}^{k}[Y_k,Z_{k-j}]\rangle_{\mathfrak{g}}.\end{aligned}\tag{2.9}$$

To see that (2.7) holds if suffices to observe that both (2.8) and (2.9) can be written as

$$\begin{aligned}\langle[X(\lambda),Y(\lambda)],Z(\lambda)\rangle_{\mathcal{A}} &= \sum_{n=0}^{\infty}a_n\sum_{j_1+j_2+j_3=n}\langle[X_{j_1},Y_{j_2}],Z_{j_3}\rangle_{\mathfrak{g}}\\ \langle X(\lambda),[Y(\lambda),Z(\lambda)]\rangle_{\mathcal{A}} &= \sum_{n=0}^{\infty}a_n\sum_{j_1+j_2+j_3=n}\langle X_{j_1},[Y_{j_2},Z_{j_3}]\rangle_{\mathfrak{g}}\end{aligned}$$

and that the equation (2.7) immediately follows using the ad–invariance of the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

$\square$

As already remarked the bilinear form (2.6) is degenerated for every sequence $\mathcal{A} = \{a_n\}_{n\in\mathbb{N}}$. It is namely easily checked that its radical is given by the subspace of particular "polynomials" in $\mathfrak{g}(\lambda)$ which vanish in $\lambda = 1$. This is due to the fact that our bilinear map can be factorized in a bilinear map from $\mathfrak{g}(\lambda) \times \mathfrak{g}(\lambda)$ to $\mathbb{C}(\lambda)$ and an "evaluation" map (in $\lambda = 1$) from $\mathbb{C}(\lambda)$ to $\mathbb{C}$. It is matter of fact that these two maps commute in an appropriate but straightforward sense, and while the first map is not degenerate the second one has of course always a non trivial kernel. For example, in the most interesting case, when $a_n = 1$ for all n, the kernel coincides with the linear space of all polynomials $X(\lambda)$ which vanish in $\lambda = 1$.

For what we have in mind the most important property of the infinite dimensional Lie algebra $\mathfrak{g}(\lambda)$ is that it has an infinite number of nested ideals. Namely for each $k \in \mathbb{N}$ let us consider the subspace $I_k(\lambda)$ of all elements of $\mathfrak{g}(\lambda)$ which can be written as the product $\lambda^{k+1}Y(\lambda)$ for some $Y(\lambda)$ in $\mathfrak{g}(\lambda)$ i.e.,

$$I_k(\lambda) = \{X(\lambda) \in g(\lambda)|\ X(\lambda) = \lambda^{k+1}Y(\lambda) \quad Y(\lambda) \in g(\lambda)\}. \tag{2.10}$$

From (2.10) it follows immediately that $I_k(\lambda)$ is for every k an ideal and that $I_h(\lambda) \subset I_k(\lambda)$ if and only if $h \geq k$. Using this property we can define our polynomials Lie algebras:

Definition 2.3 *The polynomial Lie algebra over $\mathfrak{g}$ of degree n $\mathfrak{g}^{(n)}(\lambda)$ is the quotient Lie algebra*

$$\mathfrak{g}^{(n)}(\lambda) = \mathfrak{g}(\lambda)/I_n(\lambda). \tag{2.11}$$

Since the canonical projection $\pi_n : \mathfrak{g}(\lambda) \to \mathfrak{g}^{(n)}(\lambda)$ acts on a generic element $Z(\lambda)$ of $\mathfrak{g}(\lambda)$ as

$$\pi_n(Z(\lambda)) = \pi_n(\sum_{j=0}^{\infty} Z_j\lambda^j) = \sum_{j=0}^{n} Z_j\lambda^j \tag{2.12}$$

we can identify $\mathfrak{g}^{(n)}(\lambda)$ with the vector space of all polynomials in λ with coefficients in $\mathfrak{g}$ of degree less or equal to n endowed with the Lie bracket:

$$\left[\sum_{j=0}^{n} X_j\lambda^j, \sum_{i=0}^{n} Y_i\lambda^i\right] = \sum_{k=0}^{n}\sum_{j=0}^{k}[X_j, Y_{k-j}]\lambda^k. \tag{2.13}$$

This latter formula suggests an alternative and more direct but maybe less natural definition of $\mathfrak{g}^{(n)}(\lambda)$:

Definition 2.4 *Let $\mathfrak{g}$ any finite dimensional Lie algebra, then we will denote by $\mathfrak{g}^{(n)}$ the Lie algebra isomorph as vector space to the direct sum $\mathfrak{g}^n = \oplus_{i=0}^n \mathfrak{g}_i$, $\mathfrak{g}_i \simeq \mathfrak{g}$ of $n+1$ copies of $\mathfrak{g}$ and with Lie bracket given by the relation*

$$\begin{aligned} [(X_0, \ldots, X_n), (Y_0, \ldots, Y_n)] = &\quad (Z_0, \ldots, Z_n) \\ Z_k = \textstyle\sum_{j=0}^{k}[X_j, Y_{k-j}] &\quad\quad k = 0, \ldots, n. \end{aligned} \tag{2.14}$$

We should point out here that the Lie algebras $\mathfrak{g}^{(n)}$, which is obviously equivalent to $\mathfrak{g}^{(n)}(\lambda)$, was already defined and considered in [17] (in particular definition 2.4 coincides with the definition of the polynomial Lie algebras with bracket $[\cdot,\cdot]_0$ (in their notation) contained in their proposition 4.4). From this definition one would natural define on $\mathfrak{g}^{(n)}$ the symmetric bilinear form

$$\langle (X_0,\ldots,X_n),(Y_0,\ldots,Y_n)\rangle = \sum_{i=0}^{n} \langle X_i, Y_i\rangle \tag{2.15}$$

i.e., the bilinear form canonically defined on the direct sum $\oplus_{i=0}^{n}\mathfrak{g}_i$, which is also the projection over $\mathfrak{g}^{(n)}$ of the symmetric bilinear form (2.4) defined on $\mathfrak{g}(\lambda)$ and which is usually considered in the literature [17], [6]. But clearly being this symmetric bilinear form ad–invariant for the brackets of $\oplus_{i=0}^{n}\mathfrak{g}_i$, it fails to be ad–invariant for the modified ones (2.14). Therefore one may wonder if on $\mathfrak{g}^{(n)}$ there exist symmetric non degenerated bilinear ad–invariant forms. Since for $n>0$ the Lie algebra $\mathfrak{g}^{(n)}$ is not semisimple or reductive, even if the Lie algebra $\mathfrak{g}$ has this property, as a consequence of the Cartan theorem [11] we cannot use the usual Killing form, and we must look for something else. As we shall show in a moment the bilinear forms, we are looking for, will be deduced from the symmetric ad–invariant bilinear forms defined in 2.1. Unfortunately to "project" these latters on the quotient space $\mathfrak{g}^{(n)}$ is a little more cumbersome because the inner product of two representatives of two equivalence classes of the equivalence relation induced by $I_n(\lambda)$ depends in general from the representatives themselves and not only from their classes. Nevertheless it is possible to define on $\mathfrak{g}^{(n)}$ bilinear form which can be regarded as "natural projection" on $\mathfrak{g}^{(n)}$ of the bilinear forms (2.6) defined on $\mathfrak{g}(\lambda)$. Of course these forms on $\mathfrak{g}^{(n)}$ can be also defined independently without any referring to (2.6).
Although it should be easy for the reader to guess how these bilinear forms look like, let us construct them carefully to show into details their relation with (2.6). As already pointed out the bilinear form (2.6)
$\Omega_{\mathcal{A}}(X(\lambda),Y(\lambda)) = \langle (X(\lambda),Y(\lambda)\rangle_{\mathcal{A}}$ can be factorized as follows:

$$\begin{tikzcd} \mathfrak{g}(\lambda)\times\mathfrak{g}(\lambda) \arrow[r, "\Omega_{\mathcal{A}}^{\lambda}"] \arrow[rr, bend right, "\Omega_{\mathcal{A}}"'] & \mathbb{C}(\lambda) \arrow[r, "ev"] & \mathbb{C} \end{tikzcd} \tag{2.16}$$

where $\Omega_{\mathcal{A}}^{\lambda} : \mathfrak{g}(\lambda)\times\mathfrak{g}(\lambda) \longrightarrow \mathbb{C}$ is the map:

$$X(\lambda),Y(\lambda)\in\mathfrak{g}(\lambda) \longrightarrow \Omega_{\mathcal{A}}^{\lambda}(X(\lambda),Y(\lambda)) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \langle X_k, Y_{n-k}\rangle_{\mathfrak{g}}\lambda^n \tag{2.17}$$

while $ev : \mathbb{C}(\lambda) \longrightarrow \mathbb{C}$ is simply the map which evaluate a polynomial at the point $\lambda = 1$:

$$p(\lambda)\in\mathbb{C}^{(n)}(\lambda) \mapsto ev(p(\lambda)) = p(1). \tag{2.18}$$

Now since obviously the subspace

$$\mathcal{I}_n(\lambda) = \{p(\lambda) \in \mathbb{C}|\ \exists\ q(\lambda) \in \mathbb{C} : p(\lambda) = \lambda^{n+1} q(\lambda)\}$$

is an ideal in the commutative algebra $\mathbb{C}(\lambda)$ we can consider the quotient space

$$\mathbb{C}^{(n)}(\lambda) : \mathbb{C}(\lambda)/\mathcal{I}_n(\lambda) \tag{2.19}$$

and the corresponding quotient map Π^n, which can be explicitly written as

$$\begin{aligned} \Pi^n : \mathbb{C}(\lambda) &\longrightarrow \mathbb{C}^{(n)}(\lambda) \\ \Pi^n\left(\sum_{j=0}^{\infty} p_j \lambda^j\right) &= \sum_{j=0}^{n} p_j \lambda^j. \end{aligned} \tag{2.20}$$

Then it is easily to show that it holds:

Proposition 2.5 *There exists a unique map*

$$\Omega_{\mathcal{A}}^{(n)}(\lambda) : \mathfrak{g}(\lambda) \times \mathfrak{g}(\lambda) \longrightarrow \mathbb{C}^{(n)}(\lambda)$$

which makes the diagram :

$$\begin{tikzcd} \mathfrak{g}(\lambda) \times \mathfrak{g}(\lambda) \arrow[rr, "\Omega_{\mathcal{A}}^{\lambda}"] \arrow[d, "\pi_n \times \pi_n"'] \arrow[drr, phantom, "\circlearrowleft"] & & \mathbb{C}(\lambda) \arrow[d, "\Pi^n"] \\ \mathfrak{g}^{(k)}(\lambda) \times \mathfrak{g}^{(k)}(\lambda) \arrow[rr, "\Omega_{\mathcal{A}}^{(n)}(\lambda)"'] & & \mathbb{C}^{(k)}(\lambda) \end{tikzcd} \tag{2.21}$$

commutative, where the maps $\Omega_{\mathcal{A}}^{\lambda}$, π_n and Π^n are respectively given in (2.17),(2.12) and (2.20).

Proof It is almost evident that the map

$$\begin{aligned} \Omega_{\mathcal{A}}^{(n)}(\lambda) : \mathfrak{g}^{(n)}(\lambda) \times \mathfrak{g}^{(n)}(\lambda) &\longrightarrow \mathbb{C}^{(n)}(\lambda) \\ X(\lambda), Y(\lambda) \in \mathfrak{g}^{(n)}(\lambda) &\mapsto \Omega_{\mathcal{A}}^{(n)}(\lambda)(X(\lambda), Y(\lambda)) \\ &= \sum_{j=0}^{n} a_j \sum_{i=0}^{j} \langle X_i, Y_{j-i}\rangle_{\mathfrak{g}} \lambda^j \end{aligned} \tag{2.22}$$

makes the diagram (2.21) commutative. To show the unicity it is enough to check that if $X(\lambda)$ and $Y(\lambda)$ are two elements in $\mathfrak{g}^{(n)}(\lambda)$ then for every $\widetilde{X}(\lambda)$ and $\widetilde{Y}(\lambda)$ respectively in ${\pi_n}^{-1}(X(\lambda))$ and ${\pi_n}^{-1}(Y(\lambda))$ it holds:

$$\Omega_{\mathcal{A}}^{(n)}(\lambda)(X(\lambda), Y(\lambda)) = \Pi^n(\Omega_{\mathcal{A}}^{\lambda}(\widetilde{X}(\lambda), \widetilde{Y}(\lambda))). \tag{2.23}$$

But this is a straightforward computation. Indeed any $\widetilde{X}(\lambda)$ and $\widetilde{Y}(\lambda)$ will have the form:

$$\widetilde{X}(\lambda) = X(\lambda) + \sum_{j\geq n+1} c_j\lambda^j$$
$$\widetilde{Y}(\lambda) = Y(\lambda) + \sum_{j\geq n+1} d_j\lambda^j$$

for some elements $c_j, d_j \in \mathfrak{g}$, therefore

$$\begin{aligned}\Omega^\lambda_{\mathcal{A}}(\widetilde{X}(\lambda), \widetilde{Y}(\lambda)) &= \Omega^\lambda_{\mathcal{A}}(X(\lambda), Y(\lambda)) + \Omega^\lambda_{\mathcal{A}}(X(\lambda), d(\lambda)) \\ &\quad + \Omega^\lambda_{\mathcal{A}}(c(\lambda), Y(\lambda)) + \Omega^\lambda_{\mathcal{A}}(c(\lambda), d(\lambda)),\end{aligned}$$

with

$$\begin{aligned}\Omega^\lambda_{\mathcal{A}}(X(\lambda), Y(\lambda)) &= \sum_{j=0}^{n} a_j \sum_{i=0}^{j} \langle X_i, Y_{j-i}\rangle_{\mathfrak{g}}\lambda^j \\ \Omega^\lambda_{\mathcal{A}}(X(\lambda), d(\lambda)) &= \sum_{j=n+1}^{\infty} a_j \sum_{i=0}^{j} \langle X_i, d_{j-i}\rangle_{\mathfrak{g}}\lambda^j \\ \Omega^\lambda_{\mathcal{A}}(c(\lambda), Y(\lambda)) &= \sum_{j=n+1}^{\infty} a_j \sum_{i=0}^{j} \langle c_{j-i}, Y_i\rangle_{\mathfrak{g}}\lambda^j \\ \Omega^\lambda_{\mathcal{A}}(c(\lambda), d(\lambda)) &= \sum_{j=2n+1}^{\infty} a_j \sum_{i=0}^{j} \langle c_i, d_{j-i}\rangle_{\mathfrak{g}}\lambda^j.\end{aligned}$$

Thus

$$\Pi^n(\Omega^\lambda_{\mathcal{A}}(\widetilde{X}(\lambda), \widetilde{Y}(\lambda))) = \sum_{j=0}^{n} a_j \sum_{i=0}^{j} \langle X_i, Y_{j-i}\rangle_{\mathfrak{g}}\lambda^j = \Omega^{(n)}_{\mathcal{A}}(\lambda)(X(\lambda), Y(\lambda)). \quad (2.24)$$

□

Definition 2.6 *Let us denote*

$$\Omega^{(n)}_{\mathcal{A}} : \mathfrak{g}^{(n)}(\lambda) \times \mathfrak{g}^{(n)}(\lambda) \longrightarrow \mathbb{C} \quad (2.25)$$

the composition $ev^{(n)} \circ \Omega^{(n)}_{\mathcal{A}}(\lambda)$ where $\Omega^{(n)}_{\mathcal{A}}(\lambda)$ is the map defined in proposition 2.5 and $ev^{(n)}$ is the "evaluation" in $\lambda = 1$ map on nilpotent commutative space $\mathbb{C}^{(n)}(\lambda)$ given by:

$$\begin{aligned}\mathbb{C}^{(n)}(\lambda) &\longrightarrow \mathbb{C} \\ p(\lambda) &\mapsto ev^{(n)}(p(\lambda)) = p(1).\end{aligned}$$

In formula

$$\Omega^{(n)}_{\mathcal{A}}(X(\lambda),Y(\lambda)) = \langle X(\lambda),Y(\lambda)\rangle^{(n)}_{\mathcal{A}} = \sum_{j=0}^{n} a_j \sum_{i=0}^{j} \langle X_i, Y_{j-i}\rangle_{\mathfrak{g}}. \tag{2.26}$$

Let us here stress once again that one can use definition 2.6 and the formula (2.26) without referring to the previous proposition or the bilinear form $\Omega^{\lambda}_{\mathcal{A}}$ and that obviously this bilinear form is also defined on the equivalent Lie algebra $\mathfrak{g}^{(n)}$. The maps $\Omega^{(n)}_{\mathcal{A}}$ are the bilinear forms which we was looking for,

Proposition 2.7 *For any $n \in \mathbb{N}$ and any set of complex numbers $\mathcal{A} = \{a_j\}_{j=0,\dots n}$, the map $\Omega^{(n)}_{\mathcal{A}} : \mathfrak{g}(\lambda) \times \mathfrak{g}(\lambda) \longrightarrow \mathbb{C}$ is a bilinear, ad–invariant and, if $a_n \neq 0$, not degenerate form.*

Proof That $\Omega^{(n)}_{\mathcal{A}}(\lambda)$ is a bilinear form follows immediately by its definition.
Further to prove its ad–invariance one has only to repeat *mutata mutandis* exactly the proof given to show the same property for the bilinear form $\Omega^{\lambda}_{\mathcal{A}}$ in proposition 2.2.
It remains to show that it is non-degenerated. This can be done in two different ways. First let us show it by proving that if $X(\lambda) \in \mathfrak{g}^{(n)}$ is such that $\langle X(\lambda),Y(\lambda)\rangle^{(n)}_{\mathcal{A}} = 0$ for every $Y(\lambda) \in \mathfrak{g}^{(n)}$ then $X(\lambda) = 0$. Indeed since $X(\lambda)$ has the form $X(\lambda) = \sum_{j=0}^{n} X_j \lambda^j$ its inner product with an element of the form $Y_n(\lambda) = Y_n \lambda^n$ will be:

$$\langle X(\lambda),Y_n(\lambda)\rangle^{(n)}_{\mathcal{A}} = a_n \langle X_0, Y_n\rangle_{\mathfrak{g}} \tag{2.27}$$

and since Y_n can be chosen arbitrarily in $\mathfrak{g}$, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is non degenerated on it and $a_n \neq 0$, $\langle X(\lambda),Y_n(\lambda)\rangle^{(n)}_{\mathcal{A}} = 0$ implies that $X_0 = 0$. Then by pairing $X(\lambda)$ with an element of the type $Y_{n-1}(\lambda) = Y_{n-1}\lambda^{n-1}$ we get:

$$\langle X(\lambda),Y_{n-1}(\lambda)\rangle^{(n)}_{\mathcal{A}} = a_n \langle X_1, Y_{n-1}\rangle_{\mathfrak{g}} \tag{2.28}$$

and then again $\langle X(\lambda),Y_{n-1}(\lambda)\rangle^{(n)}_{\mathcal{A}} = 0$ implies $X_1 = 0$. Repeating n times this argument we obtain step by step that each coefficient X_i is zero, proving the proposition.
Actually passing through the matricial representation of $\Omega^{(n)}_{\mathcal{A}}$ we can give a more compact proof of this fact. If we denote by ω the matricial representation of the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ defined on $\mathfrak{g}$ then it is immediately to show that $\Omega^{(k)}$ has the matricial form

$$\Omega_{\mathcal{A}}^{(n)} = \begin{pmatrix} a_0\omega & a_1\omega & a_2\omega & \dots & a_{n-1}\omega & a_n\omega \\ a_1\omega & a_2\omega & & \dots & a_n\omega & 0 \\ a_2\omega & & & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ a_{n-1}\omega & a_n\omega & 0 & \dots & 0 & 0 \\ a_n\omega & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{2.29}$$

But then a simple argument by induction shows that

$$\det(\Omega_{\mathcal{A}}^{(n)}) = {a_n}^n \det(\omega)^n \tag{2.30}$$

and therefore that $\Omega_{\mathcal{A}}^{(n)}$ is non degenerated if and only if ω is not degenerated.

□

The just proved proposition shows that the polynomial Lie algebra $\mathfrak{g}^{(n)}$ when $\mathfrak{g}$ is semisimple, are non trivial (non abelian or semisimple) examples of quadratic Lie algebras i.e., finite dimensional Lie algebras which possess a symmetric ad–invariant, non degenerated bilinear form [18] and [19]. This fact may justify the restriction in this section of our construction to the finite dimensional case.

Let us rest here for a while to reconsider Definition 2.4. This definition may be reformulated by saying that the Lie algebra $\mathfrak{g}^{(n)}$ can be viewed as the direct sum $\mathfrak{g}^n = \bigoplus_{k=0}^n \mathfrak{g}_k$ of $n+1$ copies of $\mathfrak{g}$ endowed with a maybe cumbersome Lie brackets. It is therefore natural to wonder if this latter bracket can be recovered from that canonically defined on the direct sum $\mathfrak{g}^n$ through a Lie algebra deformation or a Lie algebra (Wigner) contraction. This latter is actually the case though the contraction which must be considered turns out to be a little more complicate then those usually considered in the literature [2]. Therefore before giving the general construction let us describe the two simpler case in order to explain which kind of problems arise.

Actually for the first non trivial case (namely $n = 1$) one can still implement the usual Lie algebra construction up to a linear transformation. Since $\mathfrak{g}^1$ is the direct sum of two copies of $\mathfrak{g}$ we can consider on it a basis $X_i^{(k)}$ $k = 0, 1$ $i = 1, \dots, r = \dim(\mathfrak{g})$ such that $X_i^{(k)} \in \mathfrak{g}^k$ $i = 1, \dots, r$ for $k = 0, 1$ respectively and with respect to which the Lie bracket canonically defined on $\mathfrak{g}^1$ are given by:

$$\begin{aligned} \left[X_i^{(l)}, X_j^{(l)}\right] &= \textstyle\sum_{k=0}^r c_{ij}^k X_k^{(l)} \qquad && l = 0, 1 \quad i, j = 1, \dots r \\ \left[X_i^{(0)}, X_j^{(1)}\right] &= 0 && i, j = 1, \dots r \end{aligned} \tag{2.31}$$

i.e., with the same structure constant for each copy of $\mathfrak{g}$. Then by performing the parameter depending change of basis

$$
\begin{aligned}
Y_i^{(0)} &= X_i^{(0)} + X_k^{(1)} && i = 1, \dots, r \\
Y_i^{(1)} &= \zeta X_i^{(1)} && i = 1, \dots, r
\end{aligned}
$$

the equations (2.31) can be written (in the new basis) as

$$
\begin{aligned}
\left[Y_i^{(0)}, Y_j^{(0)}\right] &= \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(0)} && i, j = 1, \dots r \\
\left[Y_i^{(0)}, Y_j^{(l)}\right] &= \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(l)} && i, j = 1, \dots r \\
\left[Y_i^{(l)}, Y_j^{(l)}\right] &= \zeta \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(l)} && i, j = 1, \dots r.
\end{aligned}
$$

Now by computing the limit $\zeta \to 0$ in these latter equations one obtains exactly the Lie bracket previously defined on $\mathfrak{g}^{(1)}$. Unfortunately this procedure can not be directly and straightforward extended to the general case. Let us indeed examine the next case (i.e., $n = 2$). Here we consider again a basis $X_i^{(k)}$ $i = 1, \dots, r$ $k = 0, 1, 2$ $X_i^{(k)} \in \mathfrak{g}^k$ $i = 1, \dots, r$ $k = 0, 1, 2$ with respect to which the canonical Lie bracket can be written in the form

$$
\begin{aligned}
\left[X_i^{(l)}, X_j^{(l)}\right] &= \textstyle\sum_{k=0}^{r} c_{ij}^k X_k^{(l)} && l = 0, 1, 2 \quad i, j = 1, \dots r \\
\left[X_i^{(0)}, X_j^{(1)}\right] &= 0 && i, j = 1, \dots r
\end{aligned}
\tag{2.32}
$$

(again therefore with the same structure constants for each copy of $\mathfrak{g}$). The case $n = 2$ seems to suggest to perform first the (parameter depending) change of basis given by the equation

$$
\begin{aligned}
Y_i^{(0)} &= X_i^{(0)} + X_i^{(1)} && i = 1, \dots, r \\
Y_i^{(1)} &= \zeta X_i^{(0)} + \zeta^2 X_i^{(1)} && i = 1, \dots, r Y_i^{(2)} = \zeta^2 X_i^{(0)} \qquad i = 1, \dots, r
\end{aligned}
$$

With respect to this new basis the equations (2.32) becomes

$$
\begin{aligned}
\left[Y_i^{(0)}, Y_j^{(0)}\right] &= \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(0)} && i, j = 1, \dots r \\
\left[Y_i^{(1)}, Y_j^{(l)}\right] &= \textstyle\sum_{k=0}^{r} c_{ij}^k (Y_k^{(l)} + \zeta^2 Y_k^{(1)} - \zeta Y_k^{(2)}) && i, j = 1, \dots r \\
\left[Y_i^{(l)}, Y_j^{(2)}\right] &= \zeta \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(2)} && i, j = 1, \dots r \\
\left[Y_i^{(2)}, Y_j^{(2)}\right] &= \zeta^2 \textstyle\sum_{k=0}^{r} c_{ij}^k Y_k^{(2)} && i, j = 1, \dots r.
\end{aligned}
$$

here in the formula for $\left[Y_i^{(1)}, Y_j^{(l)}\right]$ appear "unwanted terms" which luckily enough in this specific case disappear when we perform the limit $\zeta \to 0$ giving namely the

Lie bracket for $\mathfrak{g}^{(2)}$. But nevertheless it is clear that to control the behaviour of such "unwanted terms" when the parameter ζ goes to zero will be the main issue of the general case. Indeed without a carefully choice of the dependence from the parameter ζ of the transformation of basis in $\mathfrak{g}^n$ one should fear that these terms may not go to zero or even explode to infinity. Fortunately there exist opportune choices of the ζ–depending basis transformation such that holds the

Proposition 2.8 *Let $\mathfrak{g}^n = \bigoplus_{i=0}^n \mathfrak{g}_i$ $\mathfrak{g}_i \simeq \mathfrak{g}$ the direct sum of $n+1$ copies of the Lie algebra $\mathfrak{g}$ and let be $\{X_i^{(l)}\}$ $l = 0,\dots,n$, $i = 1,\dots,r = \dim(\mathfrak{g})$ a basis for $\mathfrak{g}^n$ such that $X^{(l)} \in \mathfrak{g}_l$ for all $i = 1,\dots,r$ and such that with respect to this basis the canonical bracket for $\mathfrak{g}^n$ can be written as*

$$\begin{aligned} \left[X_i^{(l)}, X_j^{(l)}\right] &= \sum_{k=1}^r c_{ij}^k X_k^{(l)} && l = 0,\dots,n \quad i,j = 1,\dots,r \\ \left[X_i^{(l)}, X_j^{(m)}\right] &= 0 && l,m = 0,\dots,n, l \neq m \quad i,j = 1,\dots,r. \end{aligned} \tag{2.33}$$

Thus if we write the equations (2.33) with respect to the new basis

$$Y_i^{(l)}(\zeta) = \sum_{s=0}^{n-l} \zeta^{l2^s} X_i^{(s)} \quad l = 0,\dots,n \quad i = 1,\dots,r \tag{2.34}$$

and then perform the limit $\zeta \to 0$ we obtain the Lie bracket for the Lie algebra $\mathfrak{g}^{(n)}$ (2.14):

$$\begin{aligned} \left[Y_i^{(l)}, Y_j^{(m)}\right] &= \sum_{k=1}^r c_{ij}^k Y_k^{(l+m)} && l+m \leq n \quad i,j = 1,\dots,r \\ \left[Y_i^{(l)}, Y_j^{(m)}\right] &= 0 && l+m > n \quad i,j = 1,\dots,r \end{aligned} \tag{2.35}$$

Proof Let us first compute Lie bracket of two elements of the new basis say $Y_p^{(k)}$ and $Y_q^{(j)}$ (where we omit here and in what follow the explicit dependence from ζ) in terms of the old one:

$$[Y_p^{(k)}, Y_q^{(j)}] = \sum_{t=1}^{\dim\mathfrak{g}} \sum_{l=0}^{\mathrm{Min}(n-k,n-j)} \zeta^{(j+k)2^l} c_{pq}^t X_t^{(l)} \tag{2.36}$$

If $k + j > n$ we leave the expression (2.36) invariate, otherwise, if $k + j \leq n$, we write it as:

$$[Y_p^{(k)}, Y_q^{(j)}] = \sum_{t=1}^{\dim\mathfrak{g}} \sum_{l=0}^{n-k-j} \zeta^{(j+k)2^l} c_{pq}^t X_t^{(l)} + \sum_{t=1}^{\dim\mathfrak{g}} \sum_{l=n-k-j+1}^{\mathrm{Min}(n-k,n-j)} \zeta^{(j+k)2^l} c_{pq}^t X_t^{(l)} \tag{2.37}$$

which, using the definition of the elements $Y_p^{(k)}$, can be written as

$$[Y_p^{(k)}, Y_q^{(j)}] = \sum_{t=1}^{\dim\mathfrak{g}} c_{pq}^t Y_t^{(l)} + \sum_{t=1}^{\dim\mathfrak{g}} \sum_{l=n-k-j+1}^{\mathrm{Min}(n-k,n-j)} \zeta^{(j+k)2^l} c_{pq}^t X_t^{(l)}. \tag{2.38}$$

What we have to show is that if we perform the limit for $\zeta \to 0$ then the whole right hand of equation (2.36) and the second term in the right hand of (2.38) vanishes. Now the (2.36) can be written as:

$$[Y_p^{(k)}, Y_q^{(j)}] = \sum_{t=1}^{\dim \mathfrak{g}} \zeta^{\mathrm{Min}(j,k)2^l} c_{pq}^t Y_t^{\mathrm{Max}(k,j)} \tag{2.39}$$

and, since $k+j > n$ with $k, j \leq n$, implies $\mathrm{Min}(j,k) \geq 1$ performing the limit for $\zeta \to 0$ we get

$$[Y_p^{(k)}, Y_q^{(j)}] = 0. \tag{2.40}$$

In the case of equation (2.38), i.e. $k+j \leq n$, in order to avoid to invert completely equations (2.34) let us prove the following

Lemma 2.9 *The elements $2^{k(n-k)} X_p^{(k)}$ expressed in the basis $Y_j^{(n)}$ have the form:*

$$\zeta^{2^k(n-k)} X_p^{(k)} = \sum_{j=0}^{k} q_j^k(\zeta^{-1}) Y_p^{(n-j)} \tag{2.41}$$

where $\deg(q_0^k) = 1$ *and* $\deg(q_j^k) = 2^k - 2^j$

Proof We proceed by induction. The result for $k=0$ is immediately true for $k=1$ we have $\zeta^{2(n-1)} X^{(1)} = Y^{(n-1)} - \zeta^{-1} Y_p^{(n)}$.
So let us suppose (2.41) true for k and let us check it for $k+1$.
We have

$$Y_p^{(n-k-1)} = \sum_{l=0}^{k} \zeta^{2^l(n-k-1)} X_p^{(l)} + \zeta^{2^{k+1}(n-k-1)} X_p^{(k+1)}$$

which can be written as

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(n+1)} = Y_p^{(n-k-1)} - \sum_{l=0}^{k} \zeta^{2^l(l-k-1)} \zeta^{2^l(n-l)} X_p^{(l)},$$

then by induction

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(n+1)} = Y_p^{(n-k-1)} - \sum_{l=0}^{k} \sum_{j=0}^{l} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1}) Y_p^{(n-j)}.$$

If we commute the two sum in this last expression

$$\zeta^{2^{k+1}(n-k-1)} X_p^{(n+1)} = Y_p^{(n-k-1)} - \sum_{j=0}^{k} \left(\sum_{l=j}^{k} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1}) \right) Y_p^{(n-j)}.$$

From which we have

$$q_j^{k+1}(\zeta^{-1}) = -\sum_{l=j}^{k} \zeta^{2^l(l-k-1)} q_j^l(\zeta^{-1})$$

where $\deg(q_0^k) = 1$ and

$$\deg(q_j^{k+1}) = 2^k + \deg(q_j^k) = 2^k + 2^k - 2^j = 2^{k+1} - 2^j.$$

This conclues the proof of the lemma.

□

Now using this lemma we can estimate $\zeta^{2^l(j+k)} X_p^{(l)}$, we have indeed that

$$\begin{aligned}
\zeta^{2^l(j+k)} X_p^{(l)} &= \zeta^{2^l(j+k)+2^l(n-l-n+l)} X_p^{(l)} = \zeta^{2^l(j+k-n+l)} \zeta^{2^l(n-l)} X_p^{(l)} \\
&= \zeta^{2^l(j+k-n+l)} \sum_{j=0}^{l} q_j^k(\zeta^{-1}) Y^{(n-j)}
\end{aligned}$$

Therefore since $\deg(q_j^l) \le 2^l - 1$ and $(j+k-n+l) > 1$:

$$\lim_{\zeta\to 0} \zeta^{2^l(j+k-n+l)} q_j^k(\zeta^{-1}) = 0 \qquad \forall l \quad n-k-j+1 \le l \le \mathrm{Min}(n-k, n-j).$$

□

It is natural to wonder if our generalised Wigner's contraction allows us to obtain also the wanted ad–invariant symmetric non degenerate bilinear forms on $\mathfrak{g}^{(n)}$ (2.26) as well. This result can be actually achieved but unfortunately one has first to modify the canonical bilinear form defined on the direct sum $\mathfrak{g}^n = \oplus_{k=0}^n \mathfrak{g}$ by multiplying its entries by factors depending in a quite complicate way from the parameter ζ. More precisely it holds the

Proposition 2.10 *Let us consider on the Lie algebra given by the direct sum $\mathfrak{g}^n = \oplus_{k=0}^n \mathfrak{g}$ the canonical bilinear form induced from that defined on $\mathfrak{g}$ i.e. in formula if $X = (X_0, \ldots, X_n)$ and $Y = (Y_0, \ldots, Y_n)$ are two elements of $\mathfrak{g}^n$ then*

$$\langle X, Y\rangle^n = \sum_{k=0}^{n} \langle X_k, Y_k\rangle_{\mathfrak{g}} \tag{2.42}$$

where $\langle \cdot, \cdot\rangle_{\mathfrak{g}}$ denotes the symmetric bilinear form defined on $\mathfrak{g}$. Then there exist ζ–depending factors $d_k(\zeta)$ such that the modified inner product on $\mathfrak{g}^n$ given by

$$\langle X, Y\rangle^n(\zeta) = \sum_{k=0}^{n} d_k(\zeta) \langle X_k, Y_k\rangle_{\mathfrak{g}} \tag{2.43}$$

has the following property

$$lim_{\zeta\to 0} \langle Y_p^{(j)}(\zeta), Y_q^{(k)}(\zeta)\rangle^n(\zeta) = \langle Y_p^{(j)}, Y_q^{(k)}\rangle_{\mathcal{A}}^{(n)} \tag{2.44}$$

Proof. Let us first consider the relations (2.44) when $j = 0$ and k which running between 0 and n. This relations form a system of $n+1$ independent linear equations

$$\langle Y_p^{(0)}(\zeta), Y_q^{(k)}(\zeta)\rangle = a_k \omega_{pq} \qquad k = 0, \dots n,\ p, q = 1, \dots \dim(\mathfrak{g}) \tag{2.45}$$

where $\{\omega_{pq}\}_{p,q=1,\dots,\dim\mathfrak{g}}$ is the matricial form of the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.
We actually should simply verify the weaker assumption

$$\langle Y_p^{(0)}(\zeta), Y_q^{(k)}(\zeta)\rangle \sim a_k \omega_{pq} \quad k = 0, \dots n,\ p, q = 1, \dots, \dim(\mathfrak{g})$$

but since this does not affect our proof, let us consider the equation (2.45) instead. From this equation one can explicitly compute the coefficients $d_k(\zeta)$, $k = 0, \dots, n$, altought their expression in terms of ζ turns out to be complicate. Lakely enough we need only to know that the $d_k(\zeta)$ satisfy (2.45) and use it to to prove the following technical

Lemma 2.11 *The coefficients $d_k(\zeta)$ have the following asymptotic expansion*

$$\zeta^{(n-k)2^k} d_k(\zeta) \sim (-1)^k a_n \zeta^{-2^k+1}. \tag{2.46}$$

Proof. Let us proceed by induction from formula (2.45) we have immediately

$$\zeta^n d_0(\zeta) = a_n$$

and similarly

$$\zeta^{(n-1)2} d_1(\zeta) \sim -a_n \zeta^{-2+1}.$$

Let us suppose (2.46) true for k and let us prove it for $k+1$. Using (2.45) we have

$$\zeta^{(n-k-1)2^{k+1}} d_{k+1}(\zeta) = a_{n-k-1} - \sum_{i=0}^{k} \zeta^{(n-k-1)2^i} d_i(\zeta)$$

therefore using the induction hypothesis we obtain

$$\begin{aligned}
\zeta^{(n-k-1)2^{k+1}} d_{k+1}(\zeta) = \ & a_{n-k-1} - \textstyle\sum_{i=0}^{k} \zeta^{(i-k-1)2^i} \zeta^{2^i(n-i)} d_i(\zeta) \\
& \sim a_{n-k-1} - \textstyle\sum_{i=0}^{k} (-1)^i \zeta^{(i-k-1)2^i - 2^i + 1} a_n \\
& \sim - \textstyle\sum_{i=0}^{k} (-1)^i a_n \zeta^{2^i(i-k-2)+1} \sim (-1)^{k+1} a_n \zeta^{-2^{k+1}+1}
\end{aligned}$$

which is formula (2.46) for $k+1$ proving the claim.

$\square$

We can complete our proposition. For $j+k>n$ we have indeed

$$\langle Y_p^{(j)}(\zeta), Y_q^{(k)}(\zeta)\rangle(\zeta) = \sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} d_l(\zeta)\zeta^{(j+k)2^l}\langle X_p^{(l)}, X_q^{(l)}\rangle_{\mathfrak{g}}. \tag{2.47}$$

Using lemma 2.11 we can estimate this equation as

$$\begin{aligned}&\textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k-n+l)2^l}\zeta^{(n-l)2^l} d_l(\zeta)\\ &\sim \textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}}(-1)^l a_n \zeta^{(j+k-n+l)2^l-2^l+1} = \sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}}(-1)^l a_n\zeta^{2^l(j+k-n+l-1)+1}\end{aligned}$$

and since $j+k-n+l>0$ (being $l\geq 0$) we have that

$$\lim_{\zeta\to 0}\langle Y_p^{(k)}, Y_q^{(j)}\rangle = 0 \qquad \text{for } j+k>n$$

as wanted. If viceversa $j+k\leq n$, we have that as

$$\begin{aligned}\langle Y_p^{(k)}, Y_q^{(j)}\rangle &= \textstyle\sum_{l=0}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq}\\ &= \textstyle\sum_{l=0}^{n-k-j} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq} + \sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta)\omega_{pq}\end{aligned}$$

using (2.45) we obtain immediately that

$$\sum_{l=0}^{n-k-j} \zeta^{(j+k)2^l} d_l(\zeta) = a_{k+j},$$

while for the second summand we have, using lemma 2.11

$$\sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta) \sim \sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} a_n\zeta^{(j+k-n+l-1)2^l+1}.$$

But this implies, because $l>n-k-j$ and therefore $j+k-n+l-1\geq 0$ that the parameter ζ in the addends of the above written equation appears with powers bigger then one, and therefore that

$$\lim_{\zeta\to 0}\left(\sum_{l=n-k-j+1}^{\mathrm{Min\{n-k,n-j\}}} \zeta^{(j+k)2^l} d_l(\zeta)\right) = 0$$

and then finally that

$$\lim_{\zeta\to 0}\langle Y_p^{(k)}, Y_q^{(j)}\rangle = \omega_{pq} \qquad \text{for } j+k\leq n \quad j,k\geq 1\ p,q=1,\ldots,\dim(\mathfrak{g})$$

which concludes the proof of our proposition.

$\square$

Unfortunately both the constructions of $\mathfrak{g}^{(n)}$ so far considered are not particularly well suited for the purposes we have in mind, asking for a more handable one. Therefore the last part of this section is devoted to tackle this problem and to present a matricial realization of $\mathfrak{g}^{(n)}$.
The main result in order to achieve our task is the following

Theorem 2.12 *For every n and $\mathfrak{g}$ there is an isomorphism Φ between the polynomial Lie algebra $\mathfrak{g}^{(n)}$ and the tensor product $\mathfrak{g} \otimes \mathbb{C}^{(n)}(\lambda)$: where $\mathbb{C}^{(n)}(\lambda)$ is the commutative quotient algebra $\mathbb{C}(\lambda)/\mathcal{I}^{(n)}(\lambda)$ defined in 2.19*

Proof Let Φ be the linear map:

$$\Phi : \mathfrak{g}^{(n)} \longrightarrow \mathfrak{g} \otimes \mathbb{C}^{(n)}(\lambda)$$

defined by

$$\Phi(X_0, \dots, X_n) = \sum_{i=0}^{n} X_i \otimes \lambda^i.$$

This map is obviously a Lie algebra homomorphism, we have indeed:

$$\begin{aligned} \Phi([(X_0, \dots, X_n), (Y_0, \dots, Y_n)]) &= \Phi((Z_0, \dots, Z_n)) = \sum_{i=0}^{n} Z_i \otimes \lambda^i \\ = \sum_{i=0}^{n} \sum_{j=0}^{i} [X_j, Y_{i-j}] \otimes \lambda^i &= \left[\sum_{k=0}^{n} X_k \otimes \lambda^k, \sum_{j=0}^{n} Y_j \otimes \lambda^j \right] \\ &= [\Phi((X_0, \dots, X_n)), \Phi((Y_0, \dots, Y_n))]. \end{aligned}$$

Since $\dim(\mathfrak{g}^{(n)}) = (n+1)\dim(\mathfrak{g})$ we have only to check that $\mathrm{Ker}(\Phi) = 0$ but this follows immediately from the definition of Φ.

$\square$

On behalf of the previous theorem, it remains only to look for a true matricial representation Lie algebra $\mathbb{C}^{(n)}(\lambda)$. Let us therefore prove the

Proposition 2.13 *The map ρ given by*

$$\begin{aligned} \rho : \mathbb{C}^{(n)}(\Lambda) \longrightarrow &\quad \mathrm{End}(\mathbb{C}^{(n+1)}) \\ \rho(c_i \otimes \Lambda^i) \mapsto &\quad c_i \lambda^i \end{aligned} \tag{2.48}$$

where Λ is the $(n+1) \times (n+1)$ matrix given by

$$\Lambda = \sum_{i=0}^{n} e_{i+1,i} \tag{2.49}$$

and

$$(e_{ij})_{kr} = \begin{cases} 1 & \textit{if } i = j,\ k = r \\ 0 & \textit{otherwise} \end{cases}$$

is a ring homomorphism.

Proof We have indeed that

$$\rho(\Lambda^i)\rho(\Lambda^j) = \Lambda^{i+j} = \begin{cases} \rho(\Lambda^{i+j}) & i + j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

□

Now using together Theorem 2.12 and and proposition 2.13 we get a matricial representation of $\mathfrak{g}^{(k)}$.

Theorem 2.14 *If* $\Pi : \mathfrak{g} \longrightarrow \mathrm{Aut}(\mathbb{C}^m)$ *for some* m *is a true representation of* $\mathfrak{g}$ *then the map*

$$\widetilde{\Pi} : \mathfrak{g}^{(n)} \mapsto \mathrm{Aut}(\mathbb{C}^{m(n+1)})$$

given by

$$\begin{aligned} \widetilde{\Pi}(X_0, \ldots, X_n) &= \sum_{i=0}^{n} X_i \Lambda^i && (2.50) \\ &= \begin{pmatrix} \Pi(X_0) & 0 & 0 & 0 & 0 & 0 \\ \Pi(X_1) & \Pi(X_0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \Pi(X_{n-1}) & \vdots & \ldots & \ldots & \Pi(X_0) & 0 \\ \Pi(X_n) & \Pi(X_{n-1}) & \ldots & \ldots & \Pi(X_1) & \Pi(X_0) \end{pmatrix} \end{aligned}$$

is a true representation of $\mathfrak{g}^{(n)}$.

Proof Since we have constructed a representation of $\mathbb{C}(\lambda^n)$ we have a representation of $\mathfrak{g} \otimes \mathbb{C}(\lambda^n)$ given by:

$$\Pi \otimes \rho : \mathfrak{g} \otimes \mathbb{C}(\lambda^n) \longrightarrow \mathrm{End}(\mathbb{C}^m) \otimes \mathrm{End}(\mathbb{C}^{n+1}) \cong \mathrm{End}(\mathbb{C}^{m(n+1)})$$

To bring it on $\mathfrak{g}^{(n)}$ directly we have only to use the isomorphism $\Phi : \mathfrak{g}^{(k)} \cong \mathfrak{g} \otimes \mathbb{C}(\lambda^k)$:

$$\widetilde{\Pi} = \Pi \circ \rho \circ \Phi : \mathfrak{g}^{(k)} \longrightarrow \mathrm{Aut}(\mathbb{C}^{m(n+1)})$$

$\square$

Previously in this section we have shown that if $\mathfrak{g}$ possesses an ad–invariant bilinear non degenerate form this gives rise a ad–invariant bilinear non degenerate form on $\mathfrak{g}^{(n)}$. It is therefore natural to wonder if this latter form has a natural expression in our matricial representation. This is actually the case. We have indeed for instance when $a_k = 1$ for all k:

$$\langle (X_0, \ldots, X_n), (Y_0, \ldots, Y_n) \rangle^{(n)} = \mathrm{tr}(\widetilde{\Pi}((\mathrm{X_0, \ldots, X_n})\widetilde{\Pi}((\mathrm{Y_0, \ldots, Y_n})\mathrm{C^{(n)}}) \quad (2.51)$$

where $C^{(n)}$ is the $m(n+1) \times m(n+1)$ matrix:

$$C^{(n)} = \begin{pmatrix} \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m & \ldots & \frac{1}{3}\mathbb{I}_m & \frac{1}{2}\mathbb{I}_m & \mathbb{I}_m \\ 0 & \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m & \ldots & \frac{1}{3}\mathbb{I}_m & \frac{1}{2}\mathbb{I}_m \\ 0 & 0 & \ddots & \ddots & \ddots & \frac{1}{3}\mathbb{I}_m \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n+1}\mathbb{I}_m \end{pmatrix}$$

i.e. $C^{(n)}_{p,p+k} = \frac{1}{n+1-k}\mathbb{I}_m$, $p = 0, \ldots n-k$ and $k = 0, \ldots, n$ and $C^{(n)}_{pq} = 0$ if $q < p$, where $\mathbb{I}_m$ denotes the $m \times m$ identity matrix.

3 The affine Lie algebras

In the previous section we have constructed a class of non semisimple Lie algebras which posses an ad–invariant non degenerate symmetric bilinear form. This peculiar property suggests to investigate their affinization. To this task is devoted this third section. For the construction of the affine Lie algebras we will follows that presented by Kac in his famous book [12], with the only difference that we will end up with multidimensional central extensions, namely the affinization of the Lie algebra $\mathfrak{g}^{(n)}$ will have $n+1$ central charges.

Let us consider a polynomial Lie algebra $\mathfrak{g}^{(n)}$ where $\mathfrak{g}$ is semisimple and let us first define the corresponding loop algebra

$$\mathcal{L}(\mathfrak{g}^{(n)}) = \mathfrak{g}^{(n)} \otimes_{\mathbb{C}} \mathbb{C}(t, t^{-1}) \quad (3.1)$$

where $\mathbb{C}(t, t^{-1})$ denotes as usual the algebra of Laurent polynomials in a complex variable t. On it is defined an infinite complex Lie algebra bracket $[\cdot, \cdot]_0$ by

$$[X \otimes p, Y \otimes q]_0 = [X, Y] \otimes pq \qquad (p, q \in \mathbb{C}(t, t^{-1}); X, X \in \mathfrak{g}^{(n)}).$$

Now our non degenerated ad–invariant symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{A}}^{(n)}$ (2.1) can be extended to a $\mathbb{C}(t,t^{-1})$–valued bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{A}}^{(n)t}$ on $\mathcal{L}(\mathfrak{g}^{(n)})$ by

$$\langle X\otimes p, Y\otimes q\rangle_{\mathcal{A}}^{(n)}(t)=\langle X,Y\rangle_{\mathcal{A}}^{(n)}pq. \tag{3.2}$$

Moreover this latter expression can be used to define a symmetric non degenerated ad–invariant bilinear form on $\mathcal{L}(\mathfrak{g}^{(n)})$ as

$$\langle X\otimes p, Y\otimes q\rangle_{\mathcal{A}}^{(n)t}=\mathrm{Res}(\langle X,Y\rangle_{\mathcal{A}}^{(n)}pq) \tag{3.3}$$

where the Res is the linear functional of $\mathbb{C}(t,t^{-1})$ defined by the properties

$$\mathrm{Res}(t^{-1})=1;\qquad \mathrm{Res}(\frac{dp}{dt}=0).$$

The key point in the possessing of a non degenerated ad–invariant symmetric bilinear form is that it allows us to define a $\mathbb{C}$–valued 2–cocycle on the Lie algebra $\mathfrak{g}^{(n)}$ $\Psi:\mathcal{L}(\mathfrak{g}^{(n)})\to\mathbb{C}$ (see [12]) as

$$\Psi(X(t),Y(t))=\mathrm{Res}(\langle\frac{dX(t)}{dt},Y(t)\rangle_{\mathcal{A}}^{(n)})(t). \tag{3.4}$$

This 2–cocycle in turns allows us to extend our Lie algebra $\mathcal{L}(\mathfrak{g}^{(n)})$ by a $n+1$–dimensional center. Explicitly

$$\tilde{\mathcal{L}}(\mathfrak{g}^{(n)})=\mathcal{L}(\mathfrak{g}^{(n)})\oplus\sum_{i=0}^{n}\oplus\mathbb{C}c_i \tag{3.5}$$

with Lie bracket defined as

$$\begin{aligned}[(X_0,\dots,X_n)\otimes p,(Y_0,\dots,Y_n)\otimes q] &= [(X_0,\dots,X_n)\otimes p,(Y_0,\dots,Y_n)\otimes q]_0\\ &+\textstyle\sum_{i=0}^{n}(\sum_{j=0}^{i}\Psi(X_{i-j}\otimes p,Y_j\otimes q)c_j.\end{aligned} \tag{3.6}$$

Again, as usual, we also extend every derivation D of the Lie algebra $\mathbb{C}(t,t^{-1})$ to a derivation of the whole Lie algebra $\mathcal{L}(\mathfrak{g}^{(n)})$ by setting

$$D(X\otimes p)=X\otimes D(p). \tag{3.7}$$

In particular we denote by $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ the Lie algebra obtained by adding to $\mathcal{L}(\mathfrak{g}^{(n)})$ a derivation d which acts on $\mathfrak{g}^{(n)}$ as $t\frac{d}{dt}$ and which annihilates the central charges c_i $i=0,\dots,n$. Thus we can extend the bilinear form (3.2) to a symmetric ad–invariant bilinear form on the whole algebra $\mathcal{L}(\mathfrak{g}^{(n)})$ by setting $\langle c_i,d\rangle_{\mathcal{A}}^{(n)t}=1$, $\langle c_i,c_j\rangle_{\mathcal{A}}^{(n)t}=\langle d,d\rangle_{\mathcal{A}}^{(n)t}=0$ for $i,j=0,\dots n$. Then explicitly $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ is the vector space

$$\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})=\mathcal{L}(\mathfrak{g}^{(n)})\oplus\sum_{i=0}^{n}\oplus\mathbb{C}c_i\oplus\mathbb{C}d \tag{3.8}$$

with bracket defined as follows

$$
\begin{aligned}
&[(X_0,\dots,X_n)\otimes t^p \oplus (\textstyle\sum_{i=0}^n \nu_i c_i \oplus \mu d), (Y_0,\dots,Y_n)\otimes t^q \oplus (\sum_{i=0}^n \nu_i^1 c_i \oplus \mu^1 d)] = \\
&[(X_0,\dots,X_n),(Y_0,\dots,Y_n)]\otimes t^{p+q} + (\mu q (Y_0,\dots,Y_n)\otimes t^q - \mu^1 p (X_0,\dots,X_n)\otimes t^p) \\
&+p\delta_{p,-q}\textstyle\sum_{i=0}^n \sum_{j=0}^i a_j \langle X_{i-j}, Y_j\rangle c_i.
\end{aligned} \tag{3.9}
$$

The algebra $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ defined above is the "affine Lie algebra" which was looking for. It should be here pointed out that this algebra could be also obtained as quotient from the Lie algebra $\widehat{\mathcal{L}}(\mathfrak{g})\otimes\mathbb{C}(\lambda)$, where $\widehat{\mathcal{L}}(\mathfrak{g})$ is the usual affine non twisted Lie algebra associated with the semisimple Lie algebra $\mathfrak{g}$, using the same procedure which gives the Lie algebra $\mathfrak{g}^{(n)}$ starting from the Lie algebra $\mathfrak{g}\otimes\mathbb{C}(\lambda)$. Moreover is still true that there exists a isomorphism between the Lie algebras $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ and $\widehat{\mathcal{L}}(\mathfrak{g})\otimes\mathbb{C}^{(n)}(\lambda)$ where $\mathbb{C}^{(n)}(\lambda)$ is the commutative nilpotent Lie algebra defined in (2.19), which is the obvious extension of the corresponding isomorphism for the finite dimensional Lie algebra $\mathfrak{g}^{(n)}$ defined in 2.3. This isomorphism explains very clearly the phenomenon of the multidimensional central extension, this subspace coincides indeed with the N–dimensional tensor product $c\otimes\mathbb{C}^{(n)}(\lambda)$ where c is the unique central charge of $\widehat{\mathcal{L}}(\mathfrak{g})$.

Since the aim of the next section is the construction of the vertex operators of these algebra, which roughly speaking represent certain generating series of elements of $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ rather then individual ones, it is opportune to cast formulas (3.9) in a more compact form using the formal calculus (see for example [8] or [13] for more details).

The first step in this direction is to consider the Cartan decomposition of our semisimple complex Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \oplus \mathfrak{g}_\alpha, \tag{3.10}$$

where $\mathfrak{h}$ is an once for ever fixed Cartan subalgebra of $\mathfrak{g}$, and $\Delta = (\mathfrak{g},\mathfrak{h})$ is the corresponding the root system. Let us denote further by $\Sigma = \{\alpha_1,\dots\alpha_r\}$ a subset of simple roots in Δ with $r = \dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g})$ (again once for ever fixed) and let $\{H_{\alpha_1},\dots,H_{\alpha_r}\}$ be the corresponding set of coroots in $\mathfrak{h}$. Then it is well known [11] that $\dim(\mathfrak{g})_\alpha = 1\ \forall\alpha\in\Delta$ and that there exist non trivial element X_α in g_α such that

$$\{H_{\alpha_1},\dots,H_{\alpha_r}\}\cup\{X_\alpha\}_{\alpha\in\Delta} \tag{3.11}$$

is a basis for $\mathfrak{g}$, usually called in the literature Cartan basis [11] and [12], with

$$H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}] \qquad i = 1,\dots,r.$$

This basis in turn allows us to define a basis for the whole Lie algebra $\mathfrak{g}^{(n)}$ and its affinization $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$. They will be namely respectively for $\mathfrak{g}^{(n)}$

$$\{H^k_{\alpha_1},\dots,H^k_{\alpha_r}\}\cup\{X^k_\alpha\}_{\alpha\in\Delta} \qquad k = 0,\dots n \tag{3.12}$$

and for $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$

$$\left\{\begin{array}{ll} \{H^k_{\alpha_1}\otimes t^{m_1},\dots,H^k_{\alpha_r}\otimes t^{m_r}\}\cup\{X^k_\alpha\otimes t^{m_\alpha}\}_{\alpha\in\Delta} & k=0,\dots n \\ & m_i\in\mathbb{Z} \\ c_0,\dots c_n & \\ d & \end{array}\right. \tag{3.13}$$

The corresponding Lie bracket are for $\mathfrak{g}^{(n)}$

$$\begin{array}{ll} \left[H^k_{\alpha_i},H^j_{\alpha_s}\right] & =0 \\ \left[H^k_{\alpha_i},X^j_\alpha\right] & =\left\{\begin{array}{l} \alpha(H^k_{\alpha_i})X^{k+j}_\alpha \quad \text{if } j+k\leq n \\ 0 \text{ otherwise} \end{array}\right. \\ \left[X^k_\alpha,X^j_\beta\right] & =\left\{\begin{array}{l} N(\alpha,\beta)X^{k+j}_{\alpha+\beta} \text{ if } j+k\leq n \text{ and } \alpha+\beta\in\Delta \\ 0 \text{ otherwise} \end{array}\right. \end{array} \tag{3.14}$$

with opportune integer numbers $N(\alpha,\beta)$. While for the Lie affine algebra $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ they are

$$\begin{array}{ll} \left[H^k_{\alpha_i}\otimes t^{m_i},H^j_{\alpha_s}\otimes t^{m_s}\right] & =\langle\, H^k_{\alpha_i},H^j_{\alpha_s}\rangle^{(n)}_{\mathcal{A}}\delta_{-m_i,m_s}c_{k+j} \\ \left[H^k_{\alpha_i}\otimes t^{m_i},X^j_\alpha\otimes t^{m_\alpha}\right] & =\left\{\begin{array}{l} \alpha(H^k_{\alpha_i})X^{k+j}_\alpha\otimes t^{m_i+m_\alpha} \quad \text{if } j+k\leq n \\ 0 \text{ otherwise} \end{array}\right. \\ \left[X^k_\alpha\otimes t^{m_\alpha},X^j_\beta\otimes t^{m_\beta}\right] & =\left\{\begin{array}{l} N(\alpha,\beta)X^{k+j}_{\alpha+\beta}\otimes t^{m_\alpha+m_\beta}+\langle X^k_\alpha,X^j_\beta\rangle^{(n)}_{\mathcal{A}}\delta_{m_\alpha,-m_\beta}c_{j+k} \\ \qquad \text{if } j+k\leq n \text{ and } \alpha+\beta\in\Delta \\ 0 \text{ otherwise} \end{array}\right. \\ [d,X\otimes t^m] & =mX^k\otimes t^m \qquad \forall X\in\mathfrak{g}^{(n)} \\ [c_i,X] & =0 \qquad \forall X\in\widehat{\mathcal{L}}(\mathfrak{g}^{(n)}),j=0,\dots,n. \end{array} \tag{3.15}$$

Now we are in the position to introduce our generating series ([12],[8]):

$$\begin{array}{ll} H^k_{\alpha_i}(z) & =\sum_{n\in\mathbb{Z}}H^k_{\alpha_i}\otimes t^n z^{-n} \\ X^k_{\alpha_i}(z) & =\sum_{n\in\mathbb{Z}}X^k_{\alpha_i}\otimes t^n z^{-n} \\ D_z & =z\frac{\mathrm{d}}{\mathrm{dz}} \end{array} \tag{3.16}$$

where z is a formal variable. This formal operators allow us to cast (3.15) in the

Lemma 3.1 *The Lie brackets (3.14) for the affine algebra $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ are equivalent to:*

$$\begin{aligned}
\left[H^k_{\alpha_i}(z_1), H^j_{\alpha_s}(z_2)\right] &= -\langle H^k_{\alpha_i}, H^j_{\alpha_s}\rangle^{(n)}_{\mathcal{A}}(D_{z_1}\delta)(z_1/z_2)c_{k+j} \\
\left[H^k_{\alpha_i}(z_1), X^j_{\alpha}(z_2)\right] &= \begin{cases} \alpha(H^k_{\alpha_i})X^{k+j}_{\alpha}(z_2)\delta(z_1/z_2) & k+j \leq n \\ 0 & \text{otherwise} \end{cases} \\
\left[X^k_{\alpha}(z_1), X^j_{\beta}(z_2)\right] &= \begin{cases} N(\alpha,\beta)X^{k+j}_{\alpha+\beta}(z_2)\delta(z_1/z_2) = & \\ -\langle X^k_{\alpha}, X^j_{\beta}\rangle^{(n)}_{\mathcal{A}}(D_{z_1}\delta)(z_1/z_2)c_{k+j} & k+j \leq n \\ 0 & \text{otherwise} \end{cases} \\
[d, X(z)] &= -D_z X(z) \qquad \forall X \in \mathfrak{g}^{(n)} \\
[c_i, X(z)] &= 0 \qquad \forall X(z) \in \widehat{\mathcal{L}}(\mathfrak{g}^{(n)}), \quad i = 0, \ldots, n
\end{aligned}$$

Proof The statement follows immediately by comparing the terms with same degree in the equations.

□

Until now we did not impose any restriction on the subset of complex numbers $\{a_0, \ldots, a_n\}$, which appear in the definition of the bilinear form (2.26), but in view of their realization as vertex operator algebra on "generalised" Fock spaces we need to suppose that every a_k is different from zero which without loss of generality boils down to set $a_k = 1$ for every k.

4 Vertex algebras representations

Now we can describe the construction of the vertex operators representation of our Lie algebras $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$ in the case when $\mathfrak{g}$ is a simple complex Lie algebra. For what observed in the previous sections we can view this generalised affine Lie algebra simply as the tensor product:

$$\widehat{\mathcal{L}}(\mathfrak{g}^{(n)}) \simeq \widehat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{C}^{(n)}(\lambda). \tag{4.1}$$

This equivalence suggests of course a way to obtain a generalised vertex operators representation of $\widehat{\mathcal{L}}(\mathfrak{g}^{(n)})$, namely if $\Gamma : \widehat{\mathcal{L}}(\mathfrak{g}) \to \mathrm{End(V)}$ is a vertex operator representation for $\widehat{\mathcal{L}}(\mathfrak{g})$ and $\rho : \mathbb{C}^{(n)}(\lambda) \to \mathrm{End}(\mathbb{C}^{n+1})$ is the representation (2.50) of $\mathbb{C}^{(n)}(\lambda)$ then our "vertex operators representations" will be the tensor product of the two:

$$\begin{aligned}
\Pi : \widehat{\mathcal{L}}(\mathfrak{g}^{(n)}) &\longrightarrow \mathrm{End}(\mathrm{V} \otimes \mathbb{C}^{n+1}) \\
\Pi(X \otimes p(\lambda)) &\mapsto \Gamma(X) \otimes \rho(p(\lambda))
\end{aligned} \tag{4.2}$$

Since this is the main object of the present work let us show in what follows how this construction generalises what has been already done for the affine Lie algebras [12], [13] and [8]. In particular we are going to extend to our case what in the cited literature are called homogeneous basic representations, because in our opinion in this setting the construction of the generalised vertex operators turns out to be more transparent.

More precisely let Q be the root lattice associated with the simple Lie algebra $\mathfrak{g}$, which we suppose to be of rank l and let $\mathbb{C}(Q)$ be its group algebra, i.e., the algebra with basis e^α $\alpha \in Q$ and multiplication:

$$e^\alpha e^\beta = e^{\alpha+\beta}, \quad e^0 = 1.$$

We shall denote by $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification of Q and by

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}(t, t^{-1}) \oplus \mathbb{C}$$

the affinization of $\mathfrak{h}$, and finally by S the symmetric algebra over the space $\mathfrak{h}^{<0} = \sum_{j<0} \mathfrak{h} \otimes t^j$ (following the literature we shall write Ht^j in place of $H \otimes t^j$). Then we can define a representation π of $\widehat{\mathfrak{h}}$ on $V_Q = S \otimes \mathbb{C}(Q)$ by setting $\pi = \pi_1 \otimes \pi_2$ where π_1 acts on S as

$$\begin{aligned} \pi_1(c_0) &= I \\ \pi_1(Ht^n)(At^s) &= \begin{cases} HAt^{n-s} \text{ if } n < 0 \\ n\delta_{n,s}\langle H|A\rangle_{\mathfrak{g}} \text{ if } n \geq 0 \end{cases} \end{aligned} \tag{4.3}$$

while π_2 act on (Q) simply by

$$\pi_2(K) = 0, \qquad \pi_2(Ht^n)e^\alpha = \delta_{n,0}\langle \alpha|H\rangle_{\mathfrak{g}} e^\alpha. \tag{4.4}$$

Let now set $H_\alpha = \alpha \otimes 1$,$\alpha(n) = \pi(H_\alpha t^n)$, $H_n = \pi(Ht^n)$, and e^α the operator on V_Q of multiplication by $1 \otimes e^\alpha$ and consider the following $\mathrm{End}(V_Q)$–valued fields:

$$\begin{aligned} H(z) &= \textstyle\sum_{n\in\mathbb{Z}} H_n z^{-n-1} \\ \Gamma_\alpha(z) &= \textstyle\sum_{k=0}^{n} (\exp\left(\sum_{n\geq 1} \frac{\alpha(-n)z^n}{n}\right) (\exp\left(\sum_{n\geq 1} \frac{\alpha(n)z^{-n}}{n}\right) e^\alpha z^\alpha \end{aligned} \tag{4.5}$$

Using these notations we can prove the

Theorem 4.1 *Let $V_Q^n = \bigoplus_{i=0}^n V_Q$ be the direct sum of $n+1$ copies of V_Q then the following End(V_Q^N)–valued fields:*

$$\begin{aligned} c_k &= c\Lambda^k \quad k = 0, \ldots, n \quad c \in \mathbb{C} \\ H_\alpha^k(z) &= \textstyle\sum_{m\in\mathbb{Z}} \alpha(m) z^{-m-1} \Lambda^k = H_\alpha(z)\Lambda^k \quad k = 0, \ldots, n \\ \Gamma_\alpha^k(z) &= \textstyle\sum_{k=0}^{n} (\exp\left(\sum_{m\geq 1} \frac{\alpha(-m)z^m}{m}\right) (\exp\left(\sum_{m\geq 1} \frac{\alpha(m)z^{-m}}{m}\right) e^\alpha z^\alpha)\Lambda^k \\ &= \Gamma_\alpha(z)\Lambda^k \quad k = 0, \ldots, n \end{aligned} \tag{4.6}$$

define a vertex operator representation of the Lie algebra $\widehat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{C}^{(n)}(\lambda)$, where the matrix Λ is given by equation (2.49).

Proof As usual we need only to check that our generating series satisfies the right OPE. But this can be easily done, keeping in mind the OPE of the fields (4.5) (see [12] and [13]). We have indeed:

$$
\begin{aligned}
H^k_\alpha(z)H^j_\beta(w) = \quad & H_\alpha(z)H_\beta(w)\Lambda^k\Lambda^j = \\
& \begin{cases} \sim \frac{\langle H_\alpha | H_\beta \rangle}{(z-w)^2}\Lambda^{k+j} \text{ if } k+j \leq n \\ 0 \text{ otherwise} \end{cases} \\
H^k(z)\Gamma^j_\alpha(w) = \quad & H(z)\Gamma_\alpha(z)\Lambda^k\Lambda^j = \\
& \begin{cases} \sim \frac{\langle H, \alpha \rangle}{z-w}\Gamma_{\alpha(z)}\Lambda^{k+j} \text{ if } k+j \leq n \\ 0 \text{ otherwise} \end{cases}
\end{aligned}
\tag{4.7}
$$

In similar way

$$
\begin{aligned}
\Gamma^k_\alpha(z)\Gamma^j_\beta(w) = \quad & \Gamma_\alpha(z)\Gamma_\beta(w)\Lambda^k\Lambda^j = 0 \qquad \text{if } \alpha+\beta \notin \Delta \\
\Gamma^k_\alpha(z)\Gamma^j_\beta(w) = \quad & \Gamma_\alpha(z)\Gamma_\beta(w)\Lambda^k\Lambda^j = \\
& \begin{cases} \sim \epsilon(\alpha,\beta)\frac{\Gamma_{\alpha+\beta}}{z-w}\Lambda^{k+j} \text{ if } k+j \leq n \\ 0 \text{ otherwise} \end{cases} \quad \text{if } \alpha+\beta \in \Delta \\
\Gamma^k_\alpha(z)\Gamma^j_{-\alpha}(w) = \quad & \Gamma_\alpha(z)\Gamma_{-\alpha}(z)\Lambda^k\Lambda^j = \\
& \begin{cases} \sim \epsilon(\alpha,-\alpha)\frac{c_{k+j}}{(z-w)^2} + \frac{\alpha(w)}{z-w}\Lambda^{k+j} \qquad \text{if } k+j \leq n \\ 0 \text{ otherwise} \end{cases}
\end{aligned}
\tag{4.8}
$$

where with $\epsilon : Q \to \{\pm\}$ is as usual 2–cocycle ([13]) such that

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)+(\alpha|\alpha)(\beta|\beta)}.$$

$\square$

4.1 Generalised boson–fermion correspondence

In the next section we shall apply the theory of Kac Wakimoto [15] on order to obtain a class of coupled soliton equations. Altought this theory may be implemented using directly the vertex operators given in Theorem 4.1 even in this case there exists a generalised fermionic construction which is in our opinion

worth to be presented at least in the case in which the simple Lie algebra $\mathfrak{g}$ is of type A.

Let us first indeed define the following (generalised polynomials) Clifford algebra $\mathrm{CL}^{(n)}$.

Definition 4.2 *Let V^{n+1} denoted the infinite dimensional complex vector space generated by the elements $\{\psi_j^{(k)}, \psi_j^{*(k)}\}_{j\in\mathbb{Z}, n=0,\dots,n}$:*

$$V^{n+1} = \sum_{k=0}^{n} \left(\sum_{i\in\mathbb{Z}} \mathbb{C}\psi_i^{(k)} + \sum_{i\in\mathbb{Z}} \mathbb{C}\psi_i^{*(k)} \right)$$

and let consider on it the symmetric bilinear form $\langle\cdot,\cdot\rangle_{V^{n+1}}$ defined by the relations

$$\begin{aligned} \langle\psi_l^{(k)}, \psi_m^{(j)}\rangle_{V^{n+1}} &= \langle\psi_l^{*(k)}, \psi_m^{*(j)}\rangle_{V^{n+1}} = 0 \qquad k, j = 0, \dots, n \quad k, j \in \mathbb{Z} \\ \langle\psi_l^{(k)}, \psi_m^{*(j)}\rangle_{V^{n+1}} &= \begin{cases} \delta_{lm} & k + j \leq n \\ 0 & k + j > n. \end{cases} \end{aligned} \tag{4.9}$$

Then we call polynomial Clifford algebra of length n $CL^{(n)}$ the algebra given by

$$CL^{(n)} = T(V^{n+1})/J^{n+1} \tag{4.10}$$

where $T(V^{n+1})$ denotes the tensor algebra over V^{n+1} and J^{n+1} is the ideal generated by the elements $x \otimes y + y \otimes x - \langle x, y\rangle_{V^{n+1}}$ $x, y \in V$.

Thus it is easily checked that our definition implies that $\mathrm{CL}^{(n)}$ is the algebra generated as complex (infinite) vector space by the elements $\{\psi_j^{(k)}, \psi_j^{*(k)}\}_{j\in\mathbb{Z}, n=0,\dots,n}$ which satisfy the following relations:

$$\begin{aligned} \psi_i^{(k)}\psi_l^{(j)} + \psi_l^{(j)}\psi_i^{(k)} &= 0 \qquad \psi_i^{*(k)}\psi_l^{*(j)} + \psi_l^{*(j)}\psi_i^{*(k)} = 0 \\ \psi_i^{(k)}\psi_l^{*(j)} + \psi_l^{*(j)}\psi_i^{(k)} &= \begin{cases} \delta_{il}\Lambda^{j+k} \text{ if } j + k \leq n \\ 0 \text{ otherwise.} \end{cases} \end{aligned} \tag{4.11}$$

The algebra $\mathrm{CL}^{(n)}$ possesses a representation on the direct sum of $n + 1$ copies of infinite wedge algebras:

$$F^{(n)} = \oplus_{i=0}^{n} F^i \tag{4.12}$$

where the spaces F^i with $i = 0, \dots, n$ are copies of the infinite wedge space F generated by the semi–infinite monomials

$$\underline{i}_1 \wedge \underline{i}_2 \wedge \dots \wedge \underline{i}_j \wedge \dots$$

where the i_j are integers such that

$$i_1 > i_2 > i_3 \qquad \text{and } i_j = i_{j-1} - 1 \quad \text{for } n \text{ big enough}$$

(see [12] for more details). Every space F^i has a charge decomposition

$$F^i = \bigoplus_{m \in \mathbb{Z}} F^i_m$$

by letting

$$|m\rangle^i = (\underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \ldots)^i$$

denote the vacuum vector of charge m in F^i and F^i_m the linear space spanned by all semi–infinite monomials in F^i, which differ from $|m\rangle^i$ only at finite number of places, in the same way we can decompose $F^{(n)}$ as $F^{(n)} = \bigoplus_{m \in \mathbb{Z}} F^{(n)}_m$ where $F^{(n)}_m = \oplus_{i=0}^n F^i_m$. On $F^{(n)}$ we define respectively the following action of $\psi_i^{(k)}$ and $\psi_i^{*(k)}$:

$$\begin{aligned}
\psi_i^{(k)}((\underline{i}_1 \wedge \quad & \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\
&= \textstyle\sum_{l=0}^{n-k} \psi_i e_{l+k,l}((\underline{i}_1 \wedge \underline{i}_2 \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\
&= (\underbrace{0, \ldots, 0}_{k}, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0)^k, \ldots, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j)^{j+k}, \ldots \\
&\qquad\qquad\qquad\qquad \ldots, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{n-k})^n
\end{aligned}$$

$$\begin{aligned}
\psi_i^{*(k)}((\underline{i}_1 \wedge \quad & \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\
&= \textstyle\sum_{l=0}^{n-k} \psi_i^* e_{l+k,l}((\underline{i}_1 \wedge \underline{i}_2 \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\
&= (0, \ldots, 0, (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0)^k, \ldots (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j)^{j+k}, \ldots \\
&\qquad\qquad\qquad\qquad \ldots, (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{n-k})^n)
\end{aligned} \tag{4.13}$$

where the action of the operators ψ_j and ψ_j^* is given by the usual formula [12]:

$$\begin{aligned}
\psi_j((\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^m) &= \begin{cases} 0 \text{ if } j = i_s \text{ for some } s \\ (-1)^s(\underline{i}_1 \wedge \ldots \wedge \underline{i}_s^m \wedge \underline{i}_{i_s+1} \wedge \ldots)^m \text{ if } i_s > j > i_{s+1} \end{cases} \\
\psi_j^*((\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^m) &= \begin{cases} 0 \text{ if } j \neq i_s \text{ for all } s \\ (-1)^{s+1}(\underline{i}_1 \wedge \ldots \wedge \underline{i}_{s-1} \wedge \underline{i}_{i_s+1} \wedge \ldots)^m \text{ if } j = i_s. \end{cases}
\end{aligned} \tag{4.14}$$

A simple computation shows indeed that the operators defined above satisfies equations (4.11) giving arise to a representation of $CL^{(n)}$. It is clear that this representation is indecomposable and that the vector

$$|0\rangle = ((0 \wedge -1 \wedge -2 \wedge \ldots)^0, \underbrace{0, \ldots, 0}_{n})$$

is a cyclic vector for it (i.e., $F^{(n)} = CL^{(n)}(|0\rangle))$ which satisfies the relations

$$\psi_j^{(k)}|0\rangle = 0 \text{ for } j \leq 0, \qquad \psi_j^{*(k)}|0\rangle = 0 \text{ for } j > 0 \qquad k = 0, \ldots, n.$$

Moreover it can be also checked that their action preserves the bilinear (non degenerated but not positive definite) form on $F^{(n)}$ given by

$$\begin{aligned} \langle(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots,)^0, \quad & \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots,)^n | (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n\rangle_{F^{(n)}} \\ &= \textstyle\sum_{l=0}^n \sum_{m=0}^l \langle \underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^l | (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{l-m}\rangle \end{aligned} \tag{4.15}$$

where $\langle\cdot|\cdot\rangle$ denote the Hermitian form on F for which the canonical basis for F is orthonormal. We have indeed:

$$\begin{aligned} &\langle(\psi_i^{(k)}(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) | (\underline{i}_1 \wedge \underline{i}_2 \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n,)\rangle_{F^{(n)}} \\ &= \langle(0, \ldots, 0, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0)^k, \ldots, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{n-k})^n | \\ &\qquad\qquad\qquad (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n,)\rangle_{F^{(n)}} \\ &= \textstyle\sum_{l=0}^{n-k} \sum_{m=0}^l \langle \psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^l | (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{l-m}\rangle \\ &= \textstyle\sum_{l=0}^{n-k} \sum_{m=0}^l \langle (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^l | \psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{l-m})\rangle \\ &= \langle(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n | \psi_i^{*(k)}((\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n)\rangle_{F^{(n)}}. \end{aligned}$$

The more significative consequence of the equations (4.11) are the following commutation relations:

$$\begin{aligned} \left[\psi_i^{(l)}\psi_j^{*(m-l)}, \psi_k^{(p)}\right] &= \begin{cases} \delta_{kj}\psi_i^{(m+p)} \text{ if } m+p \leq n \\ 0 \text{ otherwise} \end{cases} \\ \left[\psi_i^{(l)}\psi_j^{*(m-l)}, \psi_k^{*(p)}\right] &= \begin{cases} -\delta_{ki}\psi_j^{*(m+p)} \text{ if } m+p \leq n \\ 0 \text{ otherwise,} \end{cases} \end{aligned} \tag{4.16}$$

which can be checked as follows

$$\begin{aligned}\left[\psi_i^{(l)}\psi_j^{*(m-l)},\psi_k^{(p)}\right] &= \psi_i^{(l)}\psi_j^{*(m-l)}\psi_k^{(p)} - \psi_k^{(p)}\psi_i^{(l)}\psi_j^{*(m-l)}\\ &= \psi_i^{(l)}\psi_j^{*(m-l)}\psi_k^{(p)} + \psi_i^{(l)}\psi_k^{(p)}\psi_j^{(m-l)*} - \psi_i^{(l)}\psi_k^{(p)}\psi_j^{(m-l)*}\\ &\quad - \psi_k^{(p)}\psi_i^{(l)}\psi_j^{*(m-l)}\\ &= \psi_i^{(l)}\delta_{jk}\textstyle\sum_{s=0}^{n-s} e_{s+j+k,s} + \psi_i^{(p)}\psi_k^{(l)}\psi_j^{(m-l)*} - \psi_k^{(p)}\psi_i^{(l)}\psi_j^{(m-l)*}\\ &\overset{(4.11)}{=} \delta_{jk}\psi_i\textstyle\sum_{r=0}^{n-r} e_{r+l,r}\sum_{s=0}^{n-s} e_{s+j+k,s}\\ &= \begin{cases}\delta_{kj}\psi_i^{(m+p)} \text{ if } m+p\le n\\ 0 \text{ otherwise}\end{cases}\end{aligned}$$

while the similar proof for the second one is left to the reader.

The importance of equations (4.16) is due to the fact that can be used to define a representation of a "polynomial" generalisation $\mathfrak{gl}_\infty^{(n)}$ of the infinite dimensional Lie algebra $\mathfrak{gl}_\infty$

Definition 4.3 *Let $\mathfrak{gl}_\infty^{(n)}$ be the infinite dimensional Lie algebra given by the tensor product*

$$\mathfrak{gl}_\infty^{(n)} = \mathfrak{gl}_\infty \otimes \mathbb{C}^{(n)}(\lambda) \tag{4.17}$$

where $\mathbb{C}^{(n)}(\lambda)$ is the nilpotent polynomial ring (2.19).

The reader will immediately recognise that this definition is nothing else that the generalisation to the infinite dimensional case of the construction of polynomial Lie algebras given in the second section. Therefore a moment's reflection shows that, recalling the definition of the algebra $\mathfrak{gl}_\infty$, we can reformulate the above definition saying that $\mathfrak{gl}_\infty^{(n)}$ is the Lie algebra given by the linear span of the basis $\{E_{ij}^k\}_{j,i\in\mathbb{Z},k=0,\dots,n}$ with Lie brackets given by the formulas:

$$\left[E_{ij}^k, E_{lm}^s\right] = \begin{cases}\delta_{jl}E_{im}^{k+s} - \delta_{im}E_{lj}^{k+s} \text{ if } k+s\le n\\ 0 \text{ otherwise.}\end{cases} \tag{4.18}$$

Similarly, starting from equations (4.11), one can view the polynomial Clifford algebra $CL^{(n)}$ as the tensor product $CL\otimes\mathbb{C}^{(n)}(\lambda)$.

Being defined our infinite dimensional polynomial Lie algebra $\mathfrak{gl}_\infty^{(n)}$ we can construct its fermionic representation on $F^{(n)}$.

Theorem 4.4 *The map $\Psi : \mathfrak{gl}_\infty^{(n)} \to \mathrm{End}(\mathrm{F}^{(n)})$ given by:*

$$\Psi(E_{ij}^k) = \frac{1}{k+1}\sum_{l=0}^{k}\psi_i^{(k-l)}\psi_j^{*(l)} \qquad i,j\in\mathbb{Z}\quad k=0,\dots,n \tag{4.19}$$

defines a representation Ψ of $\mathfrak{gl}_\infty^{(n)}$ on $F^{(n)}$.

Proof Using formulas (4.16) we have for $k+s \le n$:

$$\begin{aligned}
\left[\Psi(E^k_{ij}), \Psi(E^s_{lm})\right] = & \ \tfrac{1}{k+1}\tfrac{1}{s+1}\left[\sum_{p=0}^k \psi_i^{(k-p)}\psi_j^{*(p)}, \sum_{q=0}^s \psi_l^{(s-q)}\psi^{*(q)}\right] \\
& = \tfrac{1}{k+1}\tfrac{1}{s+1}\sum_{p=0}^k \sum_{q=0}^s \big(\left[\psi_i^{(k-p)}\psi_j^{*(p)}, \psi_l^{(s-q)}\psi^{*(q)}\right]\big) \\
& = \tfrac{1}{k+1}\tfrac{1}{s+1}\sum_{p=0}^k \sum_{q=0}^s \big(\left[\psi_i^{(k-p)}\psi_j^{*(p)}, \psi_l^{(s-q)}\right]\psi_m^{*(q)}\big) \\
& \quad + \tfrac{1}{k+1}\tfrac{1}{s+1}\sum_{p=0}^k \sum_{q=0}^s \big(\psi_i^{(k-p)}\left[\psi_j^{*(p)}, \psi_l^{(s-q)}\psi^{*(q)}\right]\big) \\
& = \tfrac{1}{s+1}\tfrac{1}{k+1}\sum_{p=0}^k \sum_{q=0}^s \big(\delta_{jl}\psi_i^{(k+s-q)}\psi_m^{*(q)} - \delta_{mi}\psi_l^{(s-q)}\psi_j^{*(k+q)}\big) \\
& = \delta_{jl}\tfrac{1}{s+1}\sum_{q=0}^s \psi_i^{(k+s-q)}\psi_m^{*(q)} - \delta_{mi}\tfrac{1}{s+1}\sum_{q=0}^k \psi_l^{(s-q)}\psi_j^{*(k+q)}\big) \\
& = \delta_{jl}\Psi(E^{k+s}_{im}) - \delta_{mi}\Psi(E^{k+s}_{lj}) = \Psi(\left[E^k_{ij}, E^s_{lm}\right])
\end{aligned}$$

since it is easily checked that $\frac{1}{s+1}\sum_{q=0}^k \psi_l^{(s-q)}\psi_j^{*(k+q)} = \Psi(E^{k+s}_{lj})$ and $\frac{1}{s+1}\sum_{q=0}^s \psi_i^{(k+s-q)}\psi_m^{*(q)} = \Psi(E^{k+s}_{im})$. While if $k+s > n$ a similar computation gives $\left[\Psi(E^k_{ij}), \Psi(E^s_{lm})\right] = 0$ as wanted.

□

Note that, while the action of $\mathrm{CL}^{(n)}$ interchanges the charges, every subspaces $F^{(n)}_m$ is left invariant by the representation Ψ. Further these latter spaces are indecomposable so that Ψ is the direct sum of its restrictions on $F^{(n)}_m$.

We are actually also interested in the corresponding group representation, despite the fact that $\mathfrak{gl}^{(n)}_\infty = \mathrm{Lie}(\mathrm{GL}_\infty \ltimes \mathfrak{gl}_\infty)$ the exponential map of this algebra lies in a bigger group, which contains $(GL_\infty \ltimes \mathfrak{gl}_\infty)$ as a proper subgroup [20].

Proposition 4.5 *Let $N^{(n)}_\infty$ be the following subset of $\mathfrak{gl}^{(n)}_\infty$:*

$$N^{(n)}_\infty = \{I + X |\ X \in \mathfrak{gl}_\infty \otimes \lambda\mathbb{C}^{(n-1)}(\lambda)\}$$

then

1. *$N^{(n)}_\infty$ is a group.*
2. *The group $G^{(n)}_\infty = GL_\infty \ltimes N^{(n)}_\infty$ is the littlest group which contains the image of the exponential map on $\mathfrak{gl}^{(n)}_\infty$.*

Proof

1. It is enough to show that for every element $Z = I + X \in N^{(n)}$ there exists (actually unique) element $Y \in \mathfrak{n}^{(n)} = \mathfrak{gl}_\infty \otimes \lambda\mathbb{C}^{(n-1)}(\lambda)$ such that $Z = \exp(Y)$. But since this algebra is also a associative ring we can compute for every X in it the expression $Y = \ln(1+X) = \sum_{k=1}^n (-1)^{k+1}\frac{X^k}{k}$ where the sum turn out to be finite because $X^m = 0$ if $m > N$. Then a direct computation shows that $I + X = \exp(Y)$.

2. The second statement follows immediately once one recognises that GL_∞ is the exponential group of the Lie algebra $\mathfrak{gl}_\infty$ ([12]) and that $\mathfrak{gl}_\infty^{(n)} = \mathfrak{gl}_\infty \ltimes \mathfrak{n}^{(n)}$.

$\square$

The representation Ψ can be exponentiated to the Lie group $G_\infty^{(n)}$. Namely if g is an element of $G_\infty^{(n)}$ of the form $g = \exp X_0$ where X_0 belongs to $\mathfrak{gl}_\infty \otimes 1 \simeq \mathfrak{gl}_\infty$ then we have the natural extension of the usual case [12]

$$\begin{aligned}\Psi(g) \quad & ((\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\ & = ((g(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0)^0, \ldots, (g(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j)^j, \ldots (g(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n)^n)\end{aligned} \tag{4.20}$$

where

$$(g(\underline{i}_1 \wedge \underline{i}_2 \ldots)^s)^s = (\sum_{j_1 > j_2 > \ldots} \det(g^{i_1, i_2, \ldots}_{j_1, j_2, \ldots})(j_1 \wedge j_2 \ldots))^s.$$

While if g is an element of $G_\infty^{(n)}$ of the form $g = \exp X_k$ with X_k which belongs to $\mathfrak{gl}_\infty \otimes \lambda^k$ with $k > 0$ then the action becomes:

$$\begin{aligned}\Psi(\exp X_k) \quad & ((\underline{i}_1 \wedge \underline{i}_2 \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^j, \ldots (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^n) \\ & = (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{k-1}, \\ & (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^k + X_k(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^0)^k, \ldots, \\ & \ldots, (\textstyle\sum_{q=0}^{[\frac{r}{k}]} \frac{1}{q!} X_k^q (\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{r-kq})^r, \ldots, \\ & \quad \ldots, ((\textstyle\sum_{q=0}^{[\frac{n}{k}]} \frac{1}{q!} X_k^q \underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{n-kq})^n\end{aligned} \tag{4.21}$$

where we have denoted with $[\frac{r}{k}]$ the integer part of $\frac{r}{k}$.

It is still possible to construct a bosonization of the representation Ψ of $\mathfrak{gl}_\infty^{(n)}$, which generalise to the our contest that already known in the literature [12], [13],[15]. To achieve this task we have first to extend the representation Ψ from $\mathfrak{gl}_\infty^{(n)}$ to $\mathfrak{a}_\infty^{(n)} = \mathfrak{a}_\infty \otimes \mathbb{C}^{(n)}$, this requires to get rid from anomalies to modify our representation Ψ by putting

$$\Psi(E^k_{ij}) = \begin{cases} \frac{1}{k+1} \sum_{l=0}^k \psi_i^{(k-l)} \psi_j^{*(l)} \text{ if } i \neq j \text{ or } i = j > 0 \\ -\frac{1}{k+1} \sum_{l=0}^k \psi_j^{*(l)} \psi_i^{(k-l)} \text{ if } i = j \leq 0. \end{cases} \tag{4.22}$$

Next we must define the subalgebra $\mathfrak{s}^{(n)}$ spanned by the elements

$$s_i^k = \sum_{j \in \mathbb{Z}} E^k_{j,j+i}, \quad \text{and } c^k \qquad k = 0, \ldots, n$$

whose Lie brackets are

$$\left[s_p^k, s_q^j\right] = \begin{cases} p\delta_{p,-q}c^{k+j} \text{ if } j+k \leq n \\ 0 \text{ otherwise.} \end{cases} \tag{4.23}$$

Using the representation Ψ this algebra is given by the free bosonic fields α_j^k:

$$\begin{aligned} \alpha_j^k &= \tfrac{1}{k+1} \textstyle\sum_{l=0}^k \sum_{i\in\mathbb{Z}} \psi_i^{(k-l)} \psi_{i+j}^{*(l)} && \text{if } j \in \mathbb{Z}\{0\}, \quad k = 0, \ldots, n \\ \alpha_0^k &= \tfrac{1}{k+1} \textstyle\sum_{l=0}^k \sum_{i>0} \psi_i^{(k-l)} \psi_i^{*(l)} - \sum_{l=0}^k \sum_{i\leq 0} \psi_i^{(k-l)} \psi_i^{*(l)} && k = 0, \ldots, n. \end{aligned} \tag{4.24}$$

We have indeed:

$$\left[\alpha_r^k, \alpha_s^j\right] = \begin{cases} r\delta_{r,-s}\Lambda^{k+j} \text{ if } k+j \leq n \\ 0 \text{ otherwise.} \end{cases} \tag{4.25}$$

Now following Kac in [12] we introduce the bosonic Fock space $B^{(n)}$ given by the direct of $n+1$ copies of the usual bosonic fock space $B = \mathbb{C}[x_1, x_2, \ldots; q, q^{-1}]$:

$$B^{(n)} = \bigoplus_{i=0}^{n} B_i \tag{4.26}$$

where B_i $i = 0, \ldots, n$ are copies of B. For our purposes it is also useful to look at this space as the tensor product between the Fock space B and an $n+1$ dimensional complex space:

$$B^{(n)} = B \otimes \mathbb{C}^{(n+1)} \tag{4.27}$$

in what follows we shall also need the decomposition in "charged subspaces" of $B^{(n)}$ given by:

$$B^{(n)} = \bigoplus_{m\in\mathbb{Z}} B_m \qquad B_m = q^m \mathbb{C}[x_1, x_2, \ldots] \otimes \mathbb{C}^m. \tag{4.28}$$

In this setting it is namely easy to define a representation $r^{B^{(n)}}$ of the Heisenberg algebra $\mathfrak{s}^{(n)}$ on $B^{(n)}$ as:

$$\begin{aligned} &\left.\begin{aligned} r^{B^{(n)}}(s_m^k) &= \tfrac{\partial}{\partial x_m} \otimes \Lambda^k \\ r^{B^{(n)}}(s_{-m}^k) &= m x_m \otimes \Lambda^k \end{aligned}\right\} \quad \text{if } m > 0 \quad k = 0, \ldots, n \\ &r^{B^{(n)}}(s_0^k) = q\tfrac{\partial}{\partial q} \otimes \Lambda^k \qquad\qquad k = 0, \ldots, n. \end{aligned} \tag{4.29}$$

It is straightforward to see that the usual isomorphism of $\mathfrak{s} = \mathfrak{s}^{(0)}$–modules $\sigma : F \simeq B$ (see [12]) can be extended to a $\mathfrak{s}^{(n)}$–modules $\sigma^n : F^{(n)} \simeq B^{(n)}$ simply by taking the direct sum of $n+1$ copies of the isomorphism σ.

Let us now introduce in the contest of the fermionic fields the generating series:

$$\psi^{(k)}(z) = \sum_{j\in\mathbb{Z}} \psi_j^{(k)} z^j \qquad \psi^{*(k)}(z) = \sum_{j\in\mathbb{Z}} \psi_j^{*(k)} z^{-j} \quad k = 0, \dots, n \tag{4.30}$$

and also the following operator

$$\Gamma_+^k(z) = \sum_{n\geq 1} \alpha_n^k \frac{z^{-n}}{n} \qquad \Gamma_-^k(z) = \sum_{n\geq 1} \alpha_{-n}^k \frac{z^n}{n} \quad k = 0, \dots, n. \tag{4.31}$$

In the bosonic picture using the tensor product these latter generating series can be also written as:

$$\Gamma_+^k(z) = \sum_{n\geq 1} \frac{z^{-n}}{n} \frac{\partial}{\partial x_n} \otimes \Lambda^k \qquad \Gamma_-^k(z) = \sum_{n\geq 1} \frac{z^n}{n} x_n \otimes \Lambda^k. \tag{4.32}$$

The isomorphism $\sigma^n : F^{(n)} \simeq B^{(n)}$ allows us to construct a generalised boson–fermionic correspondence:

Theorem 4.6 *For every $k = 0, \dots, n$ we have:*

$$\begin{aligned} \psi^{(k)}(z) &= z^{\alpha_0^k} q \Gamma_-^k(z) \Gamma_+^k(z)^{-1} \\ \psi^{*(k)}(z) &= q^{-1} z^{-\alpha_0^k} \Gamma_-^k(z)^{-1} \Gamma_+^k(z). \end{aligned} \tag{4.33}$$

Proof Let us prove only the first of equations (4.33) since a completely similar construction works for the second ones. From the equations (4.24) and (4.30) we obtain that

$$\begin{aligned} \left[\alpha_j^k, \psi^{(i)}(z)\right] &= \begin{cases} z^j \psi^{(k+i)}(z) \text{ if } i+k \leq n \\ 0 \text{ otherwise} \end{cases} \\ \left[\alpha_j^k, \psi^{*(i)}(z)\right] &= \begin{cases} -z^j \psi^{*(k+i)}(z) \text{ if } i+k \leq n \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

now using the map $\sigma^{(n)}$ we can transport these relations to $B^{(n)}$ we have indeed for $j > 0$ we have

$$\begin{aligned} \sigma^{(n)} \left[\alpha_j^k, \psi^{(i)}(z)\right] (\sigma^{(n)})^{-1} &= \left[\tfrac{\partial}{\partial x_j} \otimes \Lambda^k, \sigma^{(n)} \psi^{(i)}(z) (\sigma^{(n)})^{-1}\right] \\ &= \begin{cases} z^j \sigma^{(n)} \psi^{(k+i)}(z) (\sigma^{(n)})^{-1} \text{ if } i+k \leq n \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

while for j negative

$$\begin{aligned}\sigma^{(n)}\left[\alpha_j^k, \psi^{(i)}(z)\right](\sigma^{(n)})^{-1} &= \left[x_j \otimes \Lambda^k, \sigma^{(n)}\psi^{(i)}(z)(\sigma^{(n)})^{-1}\right] \\ &= \begin{cases} \frac{z^{-j}}{j}\sigma^{(n)}\psi^{(k+i)}(z)(\sigma^{(n)})^{-1} \text{ if } i+k \leq n \\ 0 \text{ otherwise.}\end{cases}\end{aligned}$$

Using these equations, the fact that $\psi_j^{(k)}$ can be written as $\psi_j \Lambda^k$ and lemma 14.5 of [12] we can now conclude that the operator $\sigma^{(n)}\psi_j^{(k)}(\sigma^{(n)})^{-1}$ brings the subspace $B_m^{(n)}$ in the subspace $B_{m+1}^{(n)}$ for every m and it is of the form

$$\sigma^{(n)}\psi_j^{(k)}(\sigma^{(n)})^{-1} = C_m(z)q\Gamma^k(z)$$

with

$$\Gamma^k(z) = \left\{\exp\left(\sum_{j\leq 1} z^j x_j\right)\exp\left(-\sum_{j\leq 1}\frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right)\right\} \otimes \Lambda^k$$

while the same argument used in the proof of Theorem 14.10 in [12] shows that $C_m(z) = z^{m+1}$.

□

Theorem 4.7 *The generating series for the representation* Ψ *(4.19) of* $\mathfrak{gl}_\infty^{(n)}$ *is*

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j}\Psi(E_{ij}^k) = \left(\frac{z_1}{z_2}\right)^m \frac{1}{1-\frac{z_2}{z_1}}\Gamma^k(z_1,z_2) \tag{4.34}$$

where

$$\Gamma^k(z_1,z_2) = (k+1)\exp(\sum_{p\geq 1}(z_1^p - z_2^p)x_p)\exp(\sum_{p\geq 1}\frac{z_1^{-p}-z_2^{-p}}{p}\frac{\partial}{\partial x_p})\Lambda^k. \tag{4.35}$$

Proof We observe that from (4.30) follows

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j}\Psi(E_{ij}^k) = \frac{1}{k+1}\sum_{l=0}^k \psi_i^{(k-l)}(z_1)\psi_j^{*(l)}(z_2)$$

substituting (4.33) we get

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j}\Psi(E_{ij}^k) = \frac{1}{k+1}\sum_{l=0}^k z_1^{\alpha_0^{k-l}} q\Gamma_-^{k-l}(z_1)\Gamma_+^{k-l}(z_1)^{-1}q^{-1}z_2^{-\alpha_0^l}\Gamma_-^l(z_2)^{-1}\Gamma_+^l(z_2).$$

Since it holds [12] for every $0 \le l \le k$ that

$$\Gamma_+^{k-l}(z_1)^{-1}\Gamma_-^{l}(z_2)^{-1} = \Gamma_-^{l}(z_2)^{-1}\Gamma_+^{k-l}(z_1)^{-1}\left(1-\frac{z_2}{z_1}\right)^{-1}$$

the previous equation using also (4.32) becomes

$$\begin{aligned}
&\left(1-\tfrac{z_2}{z_1}\right)^{-1}\tfrac{1}{k+1}\textstyle\sum_{l=0}^{k}\sum_{s=0}^{n-k+l} z_1^m e_{k-l+s,s}\sum_{r=0}^{n-l} z_2^{-m} e_{l+s,s}\\
&\textstyle\sum_{s=0}^{n-k+l}\exp(\sum_{p\ge1}(z_1^p x_p)e_{k-l+s,s}\sum_{r=0}^{n-l}\exp(\sum_{p\ge1}-z_2^p x_p)e_{l+r,r}\\
&\textstyle\sum_{s=0}^{n-k+l}\exp(\sum_{p\ge1}\frac{z_1^{-p}\frac{\partial}{\partial x_p}}{p})e_{k-l+s,s}\sum_{r=0}^{n-l}\exp(\sum_{p\ge1}\frac{-z_1^{-p}\frac{\partial}{\partial x_p}}{p})e_{l+r,r}
\end{aligned}$$

and finally

$$(k+1)\left(\frac{z_1}{z_2}\right)^m\left(1-\frac{z_2}{z_1}\right)^{-1}\exp(\sum_{p\ge1}(z_1^p-z_2^p)x_p)\exp(\sum_{p\ge1}\frac{z_1^{-p}-z_2^{-p}}{p}\frac{\partial}{\partial x_p})\Lambda^k$$

$\square$

5 Coupled Hirota bilinear equations

The aim of this section is to derive from the vertex operator algebras constructed in the previous one the corresponding hierarchies of Hirota bilinear equations. The key link to connect our representations with the corresponding bilinear equations are opportune homogeneous Casimir operators acting on particular tensor product of representations. Since the Lie algebras $\mathfrak{g}^{(n)}$ and the generalised Clifford algebra $\mathrm{CL}^{(n)}$ as well have an ad–invariant symmetric bilinear non degenerate form we can use them in order to define a corresponding homogeneous Casimir operator.

Definition 5.1 *1. Let X_i $i=1,\dots,\dim(\mathfrak{g})$ be a basis for $\mathfrak{g}$ we define*

$$\begin{aligned}
\Omega &= \textstyle\sum_{k,h=0}^{n}\sum_{i,j=1}^{\dim(\mathfrak{g})}\sum_{p,q\in\mathbb{Z}}\frac{1}{h+k+1}\langle X_i^k\otimes t^p, X_j^h\otimes t^q\rangle(X_i^k\otimes t^p)\otimes(X_i^k\otimes t^q)\\
&= \textstyle\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{i,j=1}^{\dim(\mathfrak{g})}\\
&\quad\left[\textstyle\sum_{p,q\in\mathbb{Z}}\frac{1}{k+1}\langle X_i^{k-l}\otimes t^p, X_j^l\otimes t^q\rangle(X_i^{k-l}\otimes t^p)\otimes(X_i^l\otimes t^q)\right]
\end{aligned}\tag{5.1}$$

2.

$$\begin{aligned}
\Omega_1 &= \textstyle\sum_{m=0}^{n}\sum_{k=0}^{n}\sum_{i,j\in\mathbb{Z}}\frac{1}{m+k+1}\langle\psi_i^{(m)},\psi_j^{*(k)}\rangle_{V^{n+1}}\psi_i^{(m)}\otimes\psi_j^{*(k)}\\
&= \textstyle\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j\in\mathbb{Z}}\frac{1}{k+1}\psi_j^{(k-l)}\otimes\psi_j^{*(l)}.
\end{aligned}\tag{5.2}$$

Observe that these Casimir operators can be defined as the canonical Casimir operators of the respectively reduced tensor product

$$\begin{aligned}&(\widehat{\mathfrak{g}} \otimes \mathbb{C}^{(n)}(\lambda)) \otimes_{\mathbb{C}^{(1)}(\lambda)} (\widehat{\mathfrak{g}} \otimes \mathbb{C}^{(n)}(\lambda))\\ &(\mathrm{CL} \otimes \mathbb{C}^{(\mathrm{n})}(\lambda)) \otimes_{\mathbb{C}^{(1)}(\lambda)} (\mathrm{CL} \otimes \mathbb{C}^{(\mathrm{n})}(\lambda))\end{aligned} \tag{5.3}$$

since for example we have

$$(\sum_{k=0}^{n} \sum_{j \in \mathbb{Z}} \psi_j \otimes \lambda^k) \otimes_{\mathbb{C}^{(1)}(\lambda)} (\sum_{k=0}^{n} \sum_{j \in \mathbb{Z}} \psi_j^* \otimes \lambda^k) = (\sum_{k=0}^{n} \sum_{j \in \mathbb{Z}} (\psi_j \otimes \psi_j^*) \otimes \lambda^k = \Omega_1.$$

This in turn suggests to consider the action of the Casimir operators on a similar modified tensor product of representation's space. The first step towards the construction of such modified tensor product is to note (as already observed in equation (4.1) for the algebras $\mathfrak{g}^{(n)}$) that the representation's spaces can be viewed as the tensor product between a infinite dimensional space V and $\mathbb{C}^{n+1}$ where the polynomial ring $\mathbb{C}^{(n)}(\lambda)$ acts. Moreover it is easily to see that this $\mathbb{C}^{(n)}(\lambda)$–module is isomorph to $\mathbb{C}^{(n)}(\lambda)$ thought as $\mathbb{C}^{(n)}(\lambda)$–module over itself. This fact allows us to decompose our representation's space as the tensor product:

$$V^{(n)}(\lambda) = V \otimes \mathbb{C}^{(n)}(\lambda). \tag{5.4}$$

Then mimicking what done in (5.3) we consider the "modified" tensor product

$$(V \otimes \mathbb{C}^{(n)}(\lambda)) \otimes_{\mathbb{C}^{(1)}} (V \otimes \mathbb{C}^{(n)}(\lambda)) \tag{5.5}$$

where the representation of our algebra still survive, because this process boils down to perform a projection with respect to an invariant subspace namely:

$$\sum_{\substack{i,j=1\\ i+j>n}}^{n} (V \otimes \mathbb{C}^{(i)}(\lambda)) \otimes ((V \otimes \mathbb{C}^{(j)}(\lambda)).$$

These will be the space where our generalised Hirota equations will live.

5.1 Coupled KP hierarchies

Let us compute these equations explicitly starting with the case of $\mathfrak{gl}_\infty^{(n)}$.

Using equations (4.16) it is possible to check that Ω_1 commute with action of $\mathfrak{gl}_\infty^{(n)}$ on $F^{(n)}(\lambda) \otimes_{\mathbb{C}^{(1)}} F^{(n)}(\lambda)$ and therefore with that of $GL_\infty^{(n)}$. But this in turn says that any element $\tau = (\tau_0, \ldots, \tau_n)$ of the orbit of $GL_\infty^{(n)}(|0\rangle, 0, \ldots, 0)$ satisfies the equation

$$\sum_{\substack{p,q=0\\ p+q\leq n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j \in \mathbb{Z}} \frac{1}{k+1} \psi_j^{(k-l)}(\tau_p) \otimes \psi_j^{*(l)}(\tau_q) = 0 \tag{5.6}$$

and to the contrary using the argument of Theorem 14.11 in [12] that holds

Lemma 5.2 *The orbit of $GL_\infty^{(n)}|0\rangle$ is the set of all nonzero solutions $\tau \in F_0^{(n)}$ of equation (5.6).*

Our generalised Hirota bilinear equation will be the bosonic version of equation (5.6). To apply to it the isomorphism $\sigma^{(n)}$ we have to write it in terms of $\psi^{(k)}(z)$ and $\psi^{*(k)}(z)$ as

$$z^0\text{–term of } \sum_{\substack{p,q=0\\ p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{1}{k+1} \psi^{(k-l)}(z)\tau_p \otimes \psi^{*(l)}(z)\tau_q = 0. \tag{5.7}$$

Then its bosonizated form is

$$\mathrm{res}_{z=0} \sum_{\substack{p,q=0\\ p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} (\exp \sum_{j\ge 1} z^j(x'_j - x''_j))(\exp - \sum_{j\ge 1} \frac{z^{-j}}{j}(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}))\tau_p(x')\tau_q(x'') = 0. \tag{5.8}$$

Introducing the new variables

$$x_j = \frac{1}{2}(x'_j + x''_j) \qquad y_j = \frac{1}{2}(x'_j - x''_j)$$

equation (5.8) becomes:

$$\mathrm{res}_{z=0} \sum_{\substack{p,q=0\\ p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} (\exp 2 \sum_{j\ge 1} z^j(y_j))(\exp - \sum_{j\ge 1} \frac{z^{-j}}{j}(\frac{\partial}{\partial y_j}))\tau_p(x+y)\tau_q(x-y) = 0.$$

This latter equation can be easily written in terms of elementary Schur polynomials as:

$$\sum_{p=0}^{k} \sum_{j\ge 0} S_j(2y) S_{j+1}(-\tilde{\partial}_y)\tau_p(x+y)\tau_{k-p}(x-y) = 0 \qquad k = 0, \dots, n \tag{5.9}$$

where as usual $\tilde{\partial}_y$ means $\left(\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \dots\right)$. Then introducing the Hirota bilinear differentiation by:

$$P(D_1, D_2, \dots)fg = P(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots)f(x_1+u_1, x_2+u_2, \dots)g(x_1-u_1, x_2-u_2, \dots)$$

and using the Taylor formula

$$\begin{aligned} P(\tilde{\partial}_y)\tau_p(x+y)\tau_q(x-y) &= P(\tilde{\partial}_u)\tau_p(x+y+u)\tau_q(x-y-u)|_{u=0} \\ &= P(\tilde{\partial}_u)\left(\exp \textstyle\sum_{j\ge 1} y_j \frac{\partial}{\partial u_j}\right)\tau_p(x+u)\tau_q(x-u)|_{u=0}. \end{aligned}$$

we can write (5.9) in the Hirota bilinear form:

$$\sum_{p=0}^{k}\sum_{j\geq 0} S_j(2y)S_{j+1}(-\tilde{D})\left(\exp\sum_{s\geq 1} y_s D_s\right)\tau_p\tau_{k-p} \quad k=0,\ldots,n. \tag{5.10}$$

(Here again as usual $\tilde{D}$ stands for $(D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots 0)$. Expanding (5.10) as a multiple Taylor series in the variables $y_1, y_2, \ldots$ we obtain that each coefficient of the series must vanish giving arise to a hierarchy infinite number of non linear partial differential equation in a Hirota bilinear form, which of course contains the celebrated KP hierarchy. Observe that $P(D_1,\ldots,D_k)\sum_{p=0}^{k}\tau_p\tau_{k-p}=0$ identically for any odd monomial $P(D_1,\ldots,D_k)$ in the Hirota operators D_k because $\sum_{p=0}^{k}\tau_p\tau_{k-p}=\sum_{p=0}^{k}\tau_{k-p}\tau_p$ for any $k=0,\ldots,n$. Therefore the first non trivial coupled Hirota equations are:

$$\begin{aligned}
&(D_1^4+3D_2^2-4D_1D_3)\tau_0\tau_0=0\\
&(D_1^4+3D_2^2-4D_1D_3)\tau_0\tau_1=0\\
&\ldots=\ldots\\
&(D_1^4+3D_2^2-4D_1D_3)(\textstyle\sum_{p=0}^{k}\tau_p\tau_{k-p}))=0\\
&\ldots=\ldots\\
&(D_1^4+3D_2^2-4D_1D_3)(\textstyle\sum_{p=0}^{n}\tau_p\tau_{n-p}))=0
\end{aligned} \tag{5.11}$$

To write this equations in the "soliton variables" we perform the change of variables $u_0=2\frac{\partial^2\log(\tau_0)}{\partial x^2}$, $u_i=\frac{\tau_i}{\tau_0}$ which generalises to our case those proposed by Hirota, Hu and Tang in [10]. In these new variables equations (5.11) read

$$\begin{cases}
\frac{3}{4}u_{0yy}-(u_{0t}-\frac{3}{2}u_0u_{0x}-\frac{1}{4}u_{0xxx})_x=0\\[2em]
u_{kxxxx}-4u_{kxt}+3u_{kyy}+6u_0u_{kxx}\\
+\Big(\sum_{j=1}^{k-1}2u_{jx}u_{(k-j)t}+2u_{jt}u_{(k-j)x}-3u_{jy}u_{(k-j)y}\\
\qquad -6u_0u_{jx}u_{(k-j)x}-2u_{jxxx}u_{(k-j)x}\\
\qquad -3u_{jxx}u_{(k-j)xx}-2u_{jx}u_{(k-j)xxx}\Big)=0 \quad k=1,\ldots n
\end{cases} \tag{5.12}$$

where $x=x_1$, $y=x_2$ and $t=x_3$. The vertex operator construction offers a canonical way to produce a class of generalised soliton solutions for these equations. Let $u_1^j,\ldots u_N^j$ $v_1^j,\ldots v_N^j$ $j=0,\ldots,n$ be some indeterminates, then using

the property of the Taylor expansions we have that written in component

$$
\begin{aligned}
&(\prod_{j=N}^{1}(\sum_{k_j=0}^{n}\Gamma(u_j^{k_j},v_j^{k_j})\Lambda^{k_j})(\tau_0(x_1,x_2,\ldots),\ldots,\tau_n(x_1,x_2,\ldots))^T)_m\\
&=\sum^{n}{}_{\substack{k_1,\ldots,k_N,s=0\\ k_1+\ldots+k_N+s=m}}\prod_{1\le i\le j\le N}\left[\frac{(u_j^{(k_j)}-u_i^{(k_i)})(v_j^{(k_j)}-v_i^{(k_i)})}{(u_j^{(k_j)}-v_i^{(k_i)})(v_j^{(k_j)}-u_i^{(k_i)})}\right.\\
&\quad\times\left(\exp\sum_{r\ge1}\sum_{l=1}^{N}((u_l^{k_l})^r-(v_l^{k_l})^r)x_r\right)\\
&\quad\quad\left.\times\,\tau_s(\ldots,x_r-\frac{1}{r}\sum_{l=1}^{N}((u_l^{k_l})^{-r}-(v_l^{k_l})^{-r}),\ldots)\right]\qquad m=0,\ldots,n.
\end{aligned}
\tag{5.13}
$$

This equation tells us that any matrix $\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k$ acts as nilpotent operator it hold indeed

Lemma 5.3 *For every s $0\le s\le n$ we have that*

$$(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k)^s(0,\ldots,\tau_r,0,\ldots)^T=0 \tag{5.14}$$

for every $\tau_r\in\mathbb{C}(x_1,x_2,\ldots)$ and every choice of u^k and v^k, if and only if $s>\left[\frac{1+\sqrt{1+8(n-r)}}{2}\right]$ where $[x]$ denotes the integer part of x. Moreover the m–th component of the vector $(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k)^s(0,\ldots,\tau_r,0,\ldots)^T$ vanishes identically if and only if $s>\left[\frac{1+\sqrt{1+8(m-r)}}{2}\right]$.

Proof If we set in formula (5.13) $u_j^{k_j}=u^{k_j}$ and $v_j^{k_j}=v^{k_j}$ then it is easily to check that $(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k)^s(0,\ldots,\tau_r,0,\ldots)^T=0$ for every $\tau_s\in\mathbb{C}(x_1,x_2,\ldots)$ and every choice of u^{k_j} and v^{k_j} if and only if $k_j=k_i$ for some k_j and k_i in each set of non positive integers $\{k_1,\ldots k_s\}$ which appears in the right hand of (5.13). Or in other words if and only if any set of non negative integers $\{k_1,\ldots,k_s\}$ such that $\sum_{i=1}^{s}k_i=n-r$ contains at least two elements which coincide. Suppose that $\{k_1,\ldots,k_s\}$ is a sequence with all elements distinct such that $\sum_{i=1}^{s}k_i=n-r$, then, since the sequence of s non negative pairwise distinct integers whose sum is the smallest is obviously $\{0,1,\ldots,N\}$, we must have that $n-r=\sum_{i=1}^{s}k_i\ge\sum_{i=0}^{s}i=\frac{s(s-1)}{2}$. Therefore equation (5.14) is identically satisfied only and only if $s>\left[\frac{1+\sqrt{1+8(n-r)}}{2}\right]$. A completely similar argument proves the second part of the lemma.

□

Using the statement of the lemma we can write the exponential map of an element of the type $\sum_{k=0}\alpha_k\Gamma(u^k,v^k)$ as

$$\exp(\sum_{k=0}\alpha_k\Gamma(u^k,v^k))=\sum_{k=0}^{n}\Lambda^k\left(\sum_{j=0}^{\left[\frac{1+\sqrt{1+8k}}{2}\right]}\frac{1}{j!}\sum_{s_1+\ldots+s_j=k}\prod_{i=1}^{j}\alpha_{s_i}\Gamma(u_{s_i},v_{s_i})\right).$$

Therefore from the lemma 5.2 follows that the N soliton solution of the polynomial KP hierarchy is

$$\begin{aligned}
&\tau_{\alpha_0^1,\ldots,\alpha_n^1,\ldots,\alpha_n^N,u_0^1,\ldots,u_n^N,v_0^1,\ldots,v_n^N}(x) \\
&= \sum_{k=0}^{n} \Lambda^k \left(\sum_{j=0}^{\left[\frac{1+\sqrt{1+8k}}{2}\right]} \frac{1}{j!} \sum_{s_1+\ldots+s_j=k} \prod_{i=1}^{j} \alpha_{s_i} \Gamma(u_{s_i}, v_{s_i}) \right) (1,0,\ldots,0)^T.
\end{aligned} \tag{5.15}$$

In particular the 1–soliton solution (again written in component) is

$$\begin{aligned}
&(\tau_{\alpha_0,\ldots,\alpha_n,u_0,\ldots,u_n,v_0,\ldots,v_n}(x))_m \\
&= \sum_{j=0}^{\left[\frac{1+\sqrt{1+8k}}{2}\right]} \frac{1}{j!} \sum_{s_1+\ldots+s_j=k} \prod_{i=1}^{j} \alpha_{s_i} \prod_{0\le i<l\le j} \frac{(u_{s_i}-u_{i_l})(v_{s_i}-v_{s_l})}{(u_{s_i}-v_{s_l})(v_{s_i}-u_{s_l})} \\
&\times \left(\exp \sum_{r\ge 1} \sum_{i=1}^{j} (u_{s_i}^r - v_{s_i}^r) x_r \right) \qquad m = 0,\ldots,n.
\end{aligned}$$

In the simplest coupled case when $n = 1$ equations (5.12) become

$$\begin{aligned}
\tfrac{3}{4} u_{0yy} - (u_{0t} - \tfrac{3}{2} u_0 u_{0x} - \tfrac{1}{4} u_{0xxx})_x &= 0 \\
u_{1xxxx} - 4u_{1xt} + 3u_{1yy} + 6u_0 u_{1xx} &= 0.
\end{aligned} \tag{5.16}$$

For this equations the 1–soliton solution with $\alpha_0 = \alpha_1 = 1$ takes the form

$$\begin{pmatrix} \tau_0 \\ \tau_1 \end{pmatrix} = \begin{pmatrix} 1 + \exp(\sum_{r\ge 1} (u_0^r - v_0^r) x_r) \\ \exp(\sum_{r\ge 1} (u_1^r - v_1^r) x_r) \\ +2\frac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)} \exp(\sum_{r\ge 1} (u_0^r - v_0^r + u_1^r - v_0^r) x_r) \end{pmatrix}.$$

Of course in the contest of the single equations (5.16) (and actually for $n = 2$) we can view the indeterminates $x_4, x_5 \ldots$ as parameters in the expression of the solution, which will explicitly depend only from the first three ones. Therefore explicitly:

$$\begin{aligned}
u_0(x,y,t) = \;& \tfrac{1}{2}(u_0 - v_0)(\cosh(\tfrac{1}{2}(u_0 - v_0)x + (u_0^2 - v_0^2)y + (u_0^3 - v_0^3)t + \gamma_0))^{-2} \\
u_1(x,y,t) = \;& \tfrac{1}{2}\left(\cosh(\tfrac{1}{2}(u_0 - v_0)x + (u_0^2 - v_0^2)y + (u_0^3 - v_0^3)t + \gamma_0)\right)^{-1} \\
& \times \{ e^{-\frac{1}{2}((u_0-v_0)x + (u_0^2-v_0^2)y + (u_0^3-v_0^3)t + \gamma_0} \\
& + 2\frac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)} e^{\frac{1}{2}((u_0-v_0)x + (u_0^2-v_0^2)y + (u_0^3-v_0^3)t + \gamma_0)} \} \\
& \times e^{\frac{1}{2}((u_1-v_1)x + (u_1^2-v_1^2)y + (u_1^3-v_1^3)t + \gamma_1)}
\end{aligned} \tag{5.17}$$

where γ_i with $i = 0, 1$ are arbitrary constants.

5.2 Coupled KdV and Boussinesq hierarchies

Similarly we may construct a generalisation of the KdV hierarchy (i.e., a coupled KdV hierarchy) by considering the principal "basic" representation of the polynomial Lie algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$. From what done in section 4 we consider the $\hat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$–module $V_Q^n = \oplus_{j=0}^n \mathbb{C}(x_1, x_3, x_5, \ldots)$ given by the formulas

$$
\begin{array}{llll}
H_j^k & = \Lambda^k \frac{\partial}{\partial x_j}, \quad H_{-j}^k & = j x_j \Lambda^k & j \in \mathbb{N}^{\mathrm{odd}}, k = 0, 1 \ldots, n \\
c_k & = \Lambda^k, \quad 2d - \frac{1}{2} A_0^k & = - \sum_{j \in \mathbb{N}^{\mathrm{odd}}} j x_j \Lambda^k \frac{\partial}{\partial x_j} & k = 0, 1 \ldots, n \\
A^k(z) & = \frac{1}{2}(\Gamma^k(z) - 1), & & k = 0, 1 \ldots, n
\end{array}
\tag{5.18}
$$

where

$$
H_{2j+1}^k = t^j \left(X_\alpha^k - t X_{-\alpha}^k \right), \quad A_{2j}^k = -t^j \left(t H_\alpha^k \right), \quad A_{2j+1}^k = t^j \left(X_\alpha^k - t X_{-\alpha}^k \right)
$$

with

$$
X_\alpha^k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \lambda^k \quad X_{-\alpha}^k = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \lambda^k \quad H_\alpha^k = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \otimes \lambda^k
$$

and finally

$$
\Gamma^k(z) = \left(\exp 2 \sum_{j \in \mathbb{N}^{\mathrm{odd}}} z^j x_j \right) \left(\exp -2 \sum_{j \in \mathbb{N}^{\mathrm{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right) \Lambda^k.
$$

Then the polynomial Hirota bilinear equation are given by

$$
\Omega(v \otimes_{\mathbb{C}^{(1)}(\lambda)} v) = \mu v \otimes_{\mathbb{C}^{(1)}(\lambda)} v,
$$

where $\mu \in \mathbb{C}$ which is equivalent to the following hierarchy of bilinear equations:

$$
\begin{array}{c}
\sum_{p=0}^k \left(\sum_{j>0} S_j(4y_1, 0, 4y_3, \ldots) S_j(-\frac{2}{1} D_1, 0, -\frac{2}{3} D_3, \ldots) - 8 \sum_{j \in \mathbb{N}^{\mathrm{odd}}} j y_j D_j \right) \\
\times \left(\exp \sum_{j \in \mathbb{N}^{\mathrm{odd}}} y_j D_j \right) \tau_p \tau_{k-p} = 0 \quad k = 0, \ldots, n.
\end{array}
\tag{5.19}
$$

Reasoning as in the previous case of the coupled KP equations we have that the first non trivial bilinear equations in the hierarchy are:

$$
\sum_{p=0}^k (-4 D_1 D_3 + D_1^4) \tau_p \tau_{k-p} = 0 \quad k = 0, \ldots, n
$$

are a generalisation of the coupled equation (12) in [10] namely by imposing the variable transformation $u_0 = 2\frac{\partial^2 \log(\tau_0)}{\partial x^2}$, $u_i = \frac{\tau_i}{\tau_0}$ $i = 1, \ldots, n$ we get:

$$\begin{cases} (u_{0t} - \frac{3}{2}u_0 u_{0x} - \frac{1}{4}u_{0xxx})_x = 0 \\ \\ u_{kxxxx} - 4u_{kxt} + 6u_0 u_{kxx} \\ + \left(\sum_{j=1}^{k-1} 2u_{jx}u_{(k-j)t} + 2u_{jt}u_{(k-j)x} \right. \\ \qquad - 6u_0 u_{jx} u_{(k-j)x} - 2u_{jxxx}u_{(k-j)x} \\ \qquad \left. - 3u_{jxx}u_{(k-j)xx} - 2u_{jx}u_{(k-j)xxx}\right) = 0 \quad k = 1, \ldots n. \end{cases} \tag{5.20}$$

In particular for $n = 1$ we have

$$\begin{aligned} &u_{0t} - \frac{3}{2}u_0 u_{0x} - \frac{1}{4}u_{0xxx} = 0 \\ &6u_0 u_{1xx} + u_{1xxxx} - 4u_{1xt} = 0. \end{aligned} \tag{5.21}$$

This latter equations make the contact with the literature [10] and [22] (more precisely setting $u = u_0$, $v = u_{1x}$ and rescaling the time $t \to -4t$ one obtains equations (2) of [22]). Further is worth to note that by taking the derivative with respect to x of the second equation and putting $v_0 = u_0$ and $v_1 = u_{1xx}$ equations (5.21) become

$$\begin{aligned} v_{0t} &= \frac{3}{2}v_0 v_{0x} + \frac{1}{4}v_{0xxx} \\ v_{1t} &= \frac{1}{4}v_{1xxx} + \frac{3}{2}v_0 v_{1x} + \frac{3}{2}v_{0x}v_1. \end{aligned}$$

These equations are bihamiltonian with respect the two Poisson tensors [1] [7]

$$\begin{aligned} P_1 &= \begin{pmatrix} \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} & 0 \\ 0 & -2\partial_x \end{pmatrix} \\ P_2 &= \begin{pmatrix} 0 & \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} \\ \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} - 2\partial_x & 2v_1\partial_x + v_{1x} \end{pmatrix} \end{aligned}$$

namely

$$\begin{pmatrix} v_{0t} \\ v_{1t} \end{pmatrix} = P_1 \begin{pmatrix} -\frac{1}{2}v_0 \\ -\frac{1}{8}v_{1xx} - \frac{3}{4}v_0 v_1 \end{pmatrix} = P_2 \begin{pmatrix} \frac{1}{2}v_1 \\ -\frac{1}{2}v_0 \end{pmatrix}.$$

Similarly they can be also written in the Lax form $\frac{dL}{dt} = [L, P]$ where

$$\left\{ \begin{array}{rcl} L & = & \begin{pmatrix} \partial_{xx} + v_0 & 0 \\ v_1 & \partial_{xx} + v_0 \end{pmatrix} \\ P & = & \begin{pmatrix} -\partial_{xxx} - \frac{3}{4}v_{0x} - \frac{3}{2}v_0\partial_x & 0 \\ -\frac{3}{4}v_{1x} - \frac{3}{2}v_1\partial_x & -\partial_{xxx} - \frac{3}{4}v_{0x} - \frac{3}{2}v_0\partial_x \end{pmatrix} \end{array} \right. .$$

Moreover analogous changes of variables lead to the Lax pairs for the other hierarchies arising from the Lie algebras $\widehat{\mathcal{L}}(\mathfrak{sl}_k^{(n)})$.

Actually, as in the standard case, these hierarchy can be recovered from the polynomial KP (5.11) by performing a reduction procedure, which amounts to eliminate the dependence from the "even" variables x_{2j} $j \in \mathbb{N}$ of the Fock space. This in turn corresponds to restrict the representation of $\mathfrak{gl}_\infty^{(n)}$ onto its subalgebras $\hat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$, giving Lie algebraic explanation of what done in the recent literature [16]. Therefore the soliton solution for the coupled KdV hierarchies can be recovered from those written for the coupled KP equations (5.15) erasing the even variables. In the particular case whereas $n = 2$ this reduction method applied to (5.17) leads to the following solutions:

$$\begin{aligned} u_0(x,y,t) = {} & \tfrac{1}{2}(u_0 - v_0)(\cosh(\tfrac{1}{2}(u_0 - v_0)x + (u_0^3 - v_0^3)t + \gamma_0))^{-2} \\ u_1(x,y,t) = {} & \tfrac{1}{2}\left(\cosh(\tfrac{1}{2}(u_0 - v_0)x + (u_0^3 - v_0^3)t + \gamma_0)\right)^{-1} \\ & \times\{e^{-\frac{1}{2}((u_0-v_0)x+(u_0^3-v_0^3)t+\gamma_0}\\ & +2\tfrac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)}e^{\frac{1}{2}((u_0-v_0)x+(u_0^3-v_0^3)t+\gamma_0)}\} \\ & \times e^{\frac{1}{2}((u_1-v_1)x+(u_1^3-v_1^3)t+\gamma_1)} \end{aligned} \tag{5.22}$$

where γ_i with $i = 0, 1$ are still arbitrary constants.

In the same way we can recovered from the coupled KP hierarchy the "coupled Boussinesq" hierarchy by erasing all the variables x_{3j} with $j \in \mathbb{N}$, which again corresponds to restrict our representation to the Lie algebra $\widehat{\mathcal{L}}(\mathfrak{sl}_3^{(n)})$. In this case the first non trivial bilinear Hirota equations:

$$(D_1^4 + 3D_2^2)(\sum_{p=0}^{n} \tau_p\tau_{n-p}) = 0 \qquad k = 0, \ldots, n.$$

In particular when $n = 1$ putting $u_0 = 2(\log(\tau_0))_{xx}$ and as usual $u_1 = \frac{\tau_1}{\tau_0}$ we get

$$3u_{0tt} + u_{0xxxx} + 6u_{0x}^2 + 6u_0u_{0xx} = 0$$

$$3u_{1tt} + u_{1xxxx} + 6u_0u_{1xx} = 0$$

where we have put $x = x_1$ $t = x_2$. The multi–soliton solutions of these equation can obviously recovered from the solutions (5.15) by erasing the variables x_{3j}.

5.3 Coupled BKP hierarchies and their first reductions

The construction presented above can be extended to simple Lie algebras, which are not of type A. In particular we would like to finish the paper by outlining briefly the case of the Lie algebras of type B. In order to construct the bilinear Hirota equations for the polynomial BKP hierarchy, we have to consider the polynomial Clifford algebra $\mathrm{CL_B^{(n)}}$ defined (like in definition 4.2) as the Clifford algebra generated by the elements $\phi_i^{(k)}$ $i \in \mathbb{Z}$ $k = 0, \dots, n$ and by the symmetric bilinear form:

$$\langle \phi_i^{(k)}, \phi_l^{(j)} \rangle = \begin{cases} (-1)^i \delta_{i,-l} \text{ if } j+k \leq n \\ 0 \text{ otherwise.} \end{cases}$$

Let V be the irreducible Verma module with highest weight vector $|0\rangle$ for the usual Clifford Lie algebra $\mathrm{CL_B}$ (i.e. $\mathrm{CL_B^{(0)}}$). The algebra $\mathrm{CL_B^{(n)}}$ acts on $V^{(n)} = \oplus_{k=0}^n V_k$, $V_k \simeq V$ for all k by the formula

$$\phi_i^{(j)}(v_0, \dots, v_n) = (\underbrace{0, \dots, 0}_{j}, \phi_i v_0, \dots, \phi_i v_{n-j}).$$

Using this action we can define for $n \in \mathbb{Z}^{\text{odd}}$ the neutral bosonic fields:

$$\beta_m^k = \frac{1}{2(k+1)} \sum_{l=0}^{k} \sum_{j \geq 1} (-1)^{j+1} \phi_j^{(k-l)} \phi_{-j-m}^{(l)}$$

which generate the associated Heisenberg algebra

$$\left[\beta_p^h, \beta_q^k\right] = \begin{cases} \frac{1}{2} p \delta_{p,-q} \Lambda^{h+k} \text{ if } h+k \leq n \\ 0 \text{ otherwise.} \end{cases}$$

Now we can define a generalised boson–fermion correspondence of type B $\sigma_B^{(n)}$: $V^{(n)} \to B^{(n)} = \oplus_{k=0}^n B_k$ where $B_k = \mathbb{C}[x_1, x_3, x_5, \dots; q]/(q^2 - \frac{1}{2})$ for all k, which nothing else that the direct sum of $n+1$ copies of the usual isomorphism [12] σ_B and therefore

$$\sigma_B^{(n)}(\underbrace{0, \dots, 0}_{k-1}, |0\rangle, 0 \dots, 0) = (\underbrace{0, \dots, 0}_{k-1}, 1, 0 \dots, 0)$$
$$\sigma_B^{(n)} \phi_0^k(|0\rangle, 0 \dots, 0) = (\underbrace{0, \dots, 0}_{k-1}, q, 0 \dots, 0)$$

and for $p \in \mathbb{N}^{\mathrm{odd}}$

$$\sigma_B^{(n)} \beta_p^k (\sigma_B^{(n)})^{-1} = \Lambda^k \frac{\partial}{\partial x_p} \qquad \sigma_B^{(n)} \beta_{-p}^k (\sigma_B^{(n)})^{-1} = \frac{1}{2} \Lambda^k p x_p.$$

Then if we introduce the neutral fermionic fields:

$$\phi^{(k)}(z) = \sum_{i \in \mathbb{Z}} \phi_i^{(k)} z^i$$

we can show (as in the case of $\mathfrak{a}_\infty^{(n)}$) that

$$\sigma_B^{(n)} \phi^{(k)}(z) (\sigma_B^{(n)})^{-1} = \Lambda^k q \exp\left(\sum_{j \in \mathbb{N}^{\mathrm{odd}}} x_j z^j \right) \exp\left(-2 \sum_{j \in \mathbb{N}^{\mathrm{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right).$$

Our aim is now to construct a fermionic representation of the infinite dimensional polynomial Lie algebra $\mathfrak{so}_\infty^{(n)} = \mathfrak{so}_\infty \otimes \mathbb{C}^{(n)}(\lambda)$ (and actually of $\mathfrak{b}_\infty^{(n)} = \mathfrak{b}_\infty \otimes \mathbb{C}^{(n)}(\lambda)$) spanned by the elements $F_{ij}^k = (-1)^j E_{ij}^k - (-1)^i E_{-j,-i}^k$ where the E_{ij}^k are the basis of $\mathfrak{gl}_\infty^{(n)}$ previously considered. Mimicking the same proof of Theorem 4.4 one can prove indeed that the following formula:

$$\rho(F_{ij}^k) = \frac{1}{k+1} \sum_{l=0}^{k} \phi_i^{(k-l)} \phi_{-j}^{(l)}$$

defines a representation of $\mathfrak{so}_\infty^{(n)}$, which can be linearly extended to a representation of $\mathfrak{b}_\infty^{(n)}$ by putting

$$\begin{aligned} \hat{\rho}(F_{ij}^k) &= \begin{cases} \frac{1}{k+1} \sum_{l=0}^k \phi_i^{(k-l)} \phi_{-j}^{(l)} \text{ if } i \neq j \text{ or } i = j > 0 \\ \frac{1}{k+1} \sum_{l=0}^k \phi_i^{(k-l)} \phi_{-j}^{(l)} - \frac{1}{2} \Lambda^k \text{ if } i = j < 0 \end{cases} \\ \hat{\rho}(c_k) &= \Lambda^k \qquad k = 0, \ldots, n. \end{aligned}$$

This representation turns out to be the direct sum of two representation defined respectively on $V_0^{(n)}$ (the even elements $V^{(n)}$) and on $V_1^{(n)}$ (the odd ones). Moreover it can be checked that the map $\sigma_B^{(n)} : V_0^{(n)} \simeq \oplus_{k=0}^n B_{k0}$ (where $B_{k0} = \mathbb{C}(x_1, x_2, x_3, \ldots)$ for all k) is a $\mathfrak{so}_\infty^{(n)}$–isomorphism between the representation $\rho_{|V_0}$ and the following vertex operator construction of the same algebra:

$$\sum_{i,j \in \mathbb{Z}} z_1^i z_2^{-j} F_{ij}^k \mapsto \frac{1}{2} \frac{1 - z_2/z_2}{1 + z_2/z_1} (\Gamma_B^k(z_1, z_2) - 1)$$

where

$$\Gamma_B^k(z_1, z_2) = (k+1) \Lambda^k \exp\left(\sum_{j \in \mathbb{N}^{\mathrm{odd}}} x_j (z_1^j + z_2^j) \right) \exp\left(-2 \sum_{j \in \mathbb{N}^{\mathrm{odd}}} \frac{z_1^{-j} - z_2^{-j}}{j} \frac{\partial}{\partial x_j} \right).$$

In order to construct the polynomial nBKP hierarchy of Hirota bilinear equation we use the operator

$$\Omega_1^B = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{l=0}^{k} \sum_{j\in\mathbb{Z}} (-1)^j \phi_j^{(k-l)} \otimes \phi_{-j}^{(l)}$$

commuting with the action of the algebra $\mathfrak{b}_\infty^{(n)}$. The equation on $V^{(n)} \otimes_{\mathbb{C}^{(1)}(\lambda)} V^{(n)}$

$$\Omega_1^B(\tau \otimes_{\mathbb{C}^{(1)}(\lambda)} \tau) = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{l=0}^{k} (-1)^j \phi_0^{(k-l)}(\tau) \otimes_{\mathbb{C}^{(1)}(\lambda)} \phi_0^{(l)}(\tau), \quad \tau \in V_0$$

transferred to $\oplus_{k=0}^n B_{k0}$ gives rise to the coupled BKP hierarchy

$$\sum_{p=0}^{k} \sum_{j\in\mathbb{N}^{\text{odd}}} S_j(2y_j) S_j(-\frac{2}{j} D_j) \left(\exp \sum_{s\in\mathbb{N}^{\text{odd}}} y_s D_s \right) \tau_p \tau_{k-p} \quad k = 0, \ldots, n. \qquad (5.23)$$

For example the first non trivial ones (which therefore can be viewed as generalisation to the B case of those written by Hirota) are the coefficients of y_6 in the expansion of (5.23):

$$\sum_{p=0}^{k} (D_1^6 - 5D_1D_3 - 5D_3^2 + D_1D_5)\tau_p\tau_{k-p} = 0 \quad k = 0, \ldots, n.$$

Performing the change of variables $w_0 = 2\frac{\partial \log(\tau_0)}{\partial x_1}$ and $w_1 = \frac{\tau_i}{\tau_0}$ $i = 1, \ldots n$ these equations become

$$(w_{0xxxxx} + 30w_{0x}w_{0xxx} - 5w_{0xxy} - 30w_{0x}w_{0y} + 60w_{0x}^3 + 9w_{0t})_x - 5w_{0yy} = 0$$

$$-5w_{1yy} + 180w_{0x}w_{1xx} + 9w_{1xt} + 30w_{0xxx}w_{1xx}$$

$$+30w_{0x}w_{1xxxx} + w_{1xxxxxx} - 30w_{0x}w_{1xy} - 30w_{0y}w_{1xx} - 5w_{1xxxy} = 0$$

where $x = x_1$ $y = x_3$ $t = x_5$. Once again from these equations by performing opportune reduction process (namely eliminating the variable $x_{(2m+1)j}$) we can obtain the coupled B_m soliton equations. In particular for $m = 1$ we get the coupled Kotera–Sawada hierarchy [24], whose first non trivial equation when $n = 1$ $w_0 = 2(\log(\tau_0))_{xx}$ and $w_1 = \frac{\tau_1}{\tau_0}$ is

$$9w_{0t} + w_{0xxxxx} + 3w_{0x}w_{0xxx} + 3w_0w_{0xxx} + 180w_0^2w_{0x} = 0$$

$$180w_0w_{1xx} + 9w_{1xt} + 30w_{0xx}w_{1xx} + 30w_0w_{1xxxx} + w_{1xxxxxx} = 0$$

where $x = x_1$ and $t = x_5$.

Of course exactly as in the non coupled case these hierarchies can be also obtained by applying our construction to the Lie "polynomial" algebras $(A_1^{(2)})^{(n)}$. While for $m = 2$ one obtains the coupled B_2 hierarchies, which again when $n = 1$ has as first non trivial equation:

$$(w_{0xxxxx} + 30w_{0x}w_{0xxx} - 5w_{0xxt} - 30w_{0x}w_{0t} + 60w_{0x}^3)_x - 5w_{0tt} = 0$$

$$-5w_{1tt} + 180w_{0x}w_{1xx} + 30w_{0xxx}w_{1xx} + 30w_{0x}w_{1xxxx}$$

$$+w_{1xxxxxx} - 30w_{0x}w_{1xt} - 30w_{0t}w_{1xx} - 5w_{1xxxt} = 0$$

where $x = x_1$ and $t = x_3$. Finally the vertex operator construction provides (as in the case of the hierarchies of type A) the multi–soliton solutions for all the hierarchies written above.

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